Chapter One

Real Sequences

Introduction

One of the basic notions of analysis is that of a sequence (finite or infinite). It is closely connected with the theory of mappings and sets, and it is as much important as sets and points are to mathematics. Therefore we consider it here.

In this chapter we will study sequences from different angle. For instance, one can define sequences as functions for they are helpful in the study of series. Series, we will see on chapter two, can be used to represent many of the differentiable functions such as polynomial, exponential, logarithmic etc. functions. A major advantage of the series representation of functions is that it allows us to evaluate integrals of the form $\int \sin\sqrt{x}dx$ and $\int e^{-x^2}dx$ and also approximate numbers such as e, π and $\sqrt{2}$.

We can also define sequences as a map whose domain consists of all positive integers (it may contain zero). Since the domain of a sequence is known to consist of positive integers, we often omit and give the range, specifying the terms a_n in order of their indices.

Also the convergence, divergence, monotonocity and boundedness properties of a sequence which helps us in the study of the upcoming chapters will be dealt on.

Unit Objectives:

On the completion of this unit, students should be able to:

- understand the definition of a sequence;
- find limit of different sequences;
- realize convergence or divergence of a sequence;
- Understand boundedness of sequences;
- Understand the idea of monotonocity in case of sequences;
- Understand the relation between convergence and boundedness in case of sequences.

1.1.Definition and Notations of sequences

Overview:

In this section, we are going to deal with the definition and notation of the sequence by considering various examples.

Section Objectives:

At the end of this subtopic, students will be able to:

- define a sequence;
- represent a sequence using the notation;

Sequences are, basically, countably many numbers arranged in an order that may or may not exhibit certain patterns. Here is the formal definition of a sequence:

Definition 1: A sequence of real numbers is a function $f: N \to R$. A sequence can be written as $f(1), f(2), f(3), \dots$. Usually, we will denote such a sequence by the symbols $\{a_j\}_{j=1}^{\infty}$ where $a_j = f(j)$.

For example, the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$ is written as $\left\{\frac{1}{j}\right\}_{j=1}^{\infty}$. Keep in mind that despite the strange notation, a sequence can be thought of as an ordinary function. In many cases that may not be the most expedient way to look at the situation. It is often easier to simply look at a sequence as a 'list' of numbers that may or may not exhibit a certain pattern.

An ordered set of numbers such as $a_1, a_2, a_3, \dots, a_n, \dots$ is called a sequence and usually designated by $\{a_n\}_{n=1}^{\infty}$. Each number a_k is called terms of the sequence. In particular the n^{th} term of the sequence is denoted by a_n.

Remark 1:

1. If the numbers $a_1, a_2, a_3, \dots, a_n, \dots$ are real numbers, then the sequence is called real sequence.

2. In the sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$,
    - a_1 is called the first term
    - a_2 is called the second term.

In general, a_n is called the n^{th} – term of the sequence.

3. Each term of a sequence has a successor and as a result it is called an infinite sequence.
4. n - does not have to start at 1. Sometimes it starts from 0 and some positive integer m.
5. The order of the elements (terms) of the sequence matters.

   Example 1: The sequence 1, 2, 3… is different from the sequence 2, 3, 1 …

We may also define a sequence as a function

Definition 2: An infinite sequence (sequence) is a function, say f, whose domain is the set of all integers greater than or equal to some integer m (usually 0 or 1). If f is an integer greater than or equal to some integer m and $f(n) = a_n$, then we express the sequence by writing its range in any of the following ways:

i. $f(m),\ f(m+1),\ f(m+2), \ldots;$

ii. $a_m, a_{m+1}, a_{m+2}, \ldots;$

iii. $\{f(n) : n \geq m\};$

iv. $\{f(n)\}_{n=m}^{\infty}$ or $\{a_n\}_{n=m}^{\infty}$

Define a function $f(n) = a_n$ for $n \geq 1$ (1)

Then the ordered set of numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ determines a sequence. As a result we normally suppress the symbol f and just write $\{a_n\}_{n=1}^{\infty}$ for the sequence defined in (1). Similarly if $f(n) = a_n$ for $n \geq m$, then we would write $\{a_n\}_{n=m}^{\infty}$ for the sequence.

Remark 2:

- We usually write a_n instead of $f(n)$ for the value of functions at the number n.
- The sequence $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ is denoted by $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$ or $\langle a_n \rangle_{n=1}^{\infty}$.
- In the sequence $\{a_n\}_{n=m}^{\infty}$, n is called the **index** and m is called the initial index.

- The symbol used for the index is immaterial, i.e.,$\{a_n\}_{n=m}^{\infty}$ and $\{a_i\}_{i=m}^{\infty}$ are the same sequence.

Example 2: List the first five terms of the sequence

a. $\left\{\frac{2n-1}{5n+2}\right\}_{n=1}^{\infty}$ **b.** $\{ 6 \}_{n=1}^{\infty}$ **c.** $\{ 1 + (0.25)^n \}_{n=0}^{\infty}$ **d.** $\left\{\frac{2n-1}{n-2}\right\}_{n=3}^{\infty}$

Solution: To find the terms of the sequence, we simply substitute the values $n = 1, 2, 3, ...$ successively for the n^{th} - term of the sequence. Consequently the first five terms are found by substituting the values $n = 1, 2, 3, 4$ or $n = 5$ in the n^{th} - term of the sequence a_n respectively.

Now we can easily comprehend that the initial index for a and b are 1 whereas the initial index for c and d are different from 1. Hence we need to first find the n^{th} – term where the domain is the set of natural numbers.

a. Since the initial index of the sequence is 1, hence the n^{th} – term of the sequence is itself, i.e. $a_n = \frac{2n-1}{5n+2}$.

The first five terms are given by: $a_1 = \frac{1}{7}$, $a_2 = \frac{3}{12}$, $a_3 = \frac{5}{17}$, $a_4 = \frac{7}{22}$ and $a_5 = \frac{9}{27}$ obtained by substituting $n = 1, 2, 3, 4, 5$ respectively

Sequence	n^{th} – term of a_n	The first five terms
b. $\{ 6 \}_{n=1}^{\infty}$	$a_n = 6$	$6, 6, 6, 6, 6$
c. $\{ 1 + (0.25)^n \}_{n=0}^{\infty}$	$a_n = 1 + (0.25)^{n-1}$	2 , 1.25 ,1.063,1.016 ,1.004
d. $\left\{\frac{2n-1}{n-2}\right\}_{n=3}^{\infty}$	$a_n = \frac{2n+3}{n}$	5 , 3.5 , 3 ,2.75 , 2.6

Remark 3:

Sometimes a sequence is given and we may be asked to find a defined formula, however this is not always an easy task. Unless the given sequence has some kind of pattern, it won't be easy to find the defined formula.

Example 3: Find a formula for the general term a_n of the sequence

$$\frac{2}{5}, \frac{3}{25}, \frac{4}{125}, \frac{5}{625} \dots$$

assuming that the pattern of the first few terms continues.

Solution. We are given that. $a_1 = \frac{2}{5}$, $a_2 = \frac{3}{25}$, $a_3 = \frac{4}{125}$ and $a_4 = \frac{5}{625}$

Notice that the numerator of these fractions start with 2 and increases by 1 as we go to the next, that is,

The second term has numerator 3 i.e. 2 + 1

The third term has numerator 4 i.e. 3+1

…

In general, the $n^{th} -$ term has (n+1) numerator .

Clearly the denominators are powers of 5, hence the n^{th} term has 5^n denominator .

Therefore, $a_n = \frac{n+1}{5^n}$.

Caution.

- A sequence does not have to be defined by a sensible formula.

Example 4: Given the sequence 3.14 , 3.141 , 3.1415, 3.14159, 3.141592 , … where we cannot give, unless an agreement is reached, by a sensible formula.

- It is not as such trivial to get defined formula (general term) for any given sequence.

Example 5: Here are some sequences that do not have a simple defined formula

(i). { 7 , 1 , 8 , 2 , 8 , 1 , 8 , 2 , 8 , 4 , 5 , . . . }

(ii). The Fibonacci sequence $\{ f_n \}$ is defined recursively by the conditions:

$f_1 = 1, f_2 = 1,\ f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. As we can see each term is the sum of the two preceding terms and cannot be expressed in terms of only n.

The first few terms are $1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$

Exercises 1.1

1. *List the first six terms of the sequence defined by* $a_n = \frac{n}{2n+1}$ *. Does the sequence appear to have a limit? If so find it?*
2. *Find the first six terms of the sequence of numbers with general term:*
   - ***a)*** $u_n = 2$
   - ***b)*** $u_m = (-1)^m$
   - ***c)*** $u_n = n^2 - 1$
3. *Find a formula for the general term* a_n *of the sequence, assuming that the pattern of the first few terms continues.*
   - ***(i).*** $\left\{1, \frac{-2}{3}, \frac{4}{9}, \frac{-8}{7}, \ldots\right\}$
   - ***(ii).*** $\{2,7,12,17, \ldots\}$
   - ***(iii).*** $\{5,1,5,1,5,1, \ldots\}$
   - ***(iv).*** $\frac{1}{2.3}, \frac{-8}{3.4}, \frac{27}{4.5}, \frac{-64}{5.6}, \frac{125}{6.7}, \ldots$

1.2. Convergence of a sequence

Overview

In this subsection we will learn the convergence and divergence of a sequence. We also define what we it meant by a convergent sequence, and then start to apply the definition in solving different mathematical problems.

Section Objectives:

On the completion of this lesson, students will have to:

- define convergent and divergent sequences.
- distinguish convergent and divergent sequences.

A sequence $\{a_n\}_{n=1}^{\infty}$ may have the property that as n increases, then the n^{th} - term of the sequence, a_n gets very close to some real number, say L . For instance consider the sequence $\left\{\frac{1}{2n+1}\right\}_{n=1}^{\infty}$, then the sequence can be made very close to zero by choosing n sufficiently large. Whereas, some sequences may have the property that as n increases, the n^{th} - term of the sequence increases too, i.e., a_n gets to infinity. For instance consider the sequence $\{n\}_{n=1}^{\infty}$, then the sequence goes to infinity as n gets large and large real number. The following will give us the general definition for the aforementioned once.

Definition 3(Convergent Sequence): A sequence $\{a_n\}_{n=1}^{\infty}$ is said to converge into a real number L or a number L is the limit of the sequence $\{a_n\}_{n=1}^{\infty}$ if for every $\varepsilon > 0$, there exists a positive integer $\mathbb{N}$ such that if $n > \mathbb{N}$, then $|a_n - L| < \varepsilon$.

In this case we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$.

If such a number L does not exist, the sequence has no limit or diverges.

Definition 4(Divergence of a sequence) : A sequence $\{a_n\}_{n=1}^{\infty}$ is said to diverge to

(i). ∞ if for every positive number M , there is an positive integer N such that if $n > N$, then $a_n > M$, and write it as $\lim_{n\to\infty} a_n = \infty$.

(ii). $-\infty$ if for every positive number M, there is a positive integer N such that if $n > N$, then $a_n < M$. We write it as $\lim_{n\to\infty} a_n = -\infty$.

Remark 4:

i. If $\lim_{n\to\infty} a_n = L$ exists, we say that $\{a_n\}_{n=1}^{\infty}$ converges (converges to L). If such a number L does not exist, we say that the sequence $\{a_n\}_{n=1}^{\infty}$ diverges or that $\lim_{n\to\infty} a_n$ does not exist.

ii. The phrase 'convergent sequence' is used for a sequence whose limit is finite. A sequence with an infinite limit is said to diverge. There are, of course, divergent sequences that do not have infinite limits.

Example 6: $a_n = (-1)^n$ is divergent but does not have an infinite limit though its limit does not exist yet.

iii. A sequence $\{a_n\}$ which converges to zero is called **null sequence**.

Theorem 1: The limit of a sequence, if it exists, is unique.

Proof. Let $\{a_n\}$ be a convergent sequence of real numbers such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} a_n = M$.

To prove: $L = M$.

Let $\varepsilon > 0$ be given.

Since $\lim_{n\to\infty} a_n = L$, then $\exists N_1$ such that $n \geq N_1 \implies |a_n - L| < \varepsilon/2$ and

Since $\lim_{n\to\infty} a_n = M$, then $\exists N_2$ such that $n \geq N_2 \implies |a_n - M| < \varepsilon/2$

Let $N = \max\{N_1, N_2\}$

If $n \geq N$, then we have

$$|a_n - L| < \varepsilon/2 \quad \& \quad |a_n - M| < \varepsilon/2$$

Now $|L - M| = |(L - a_n) + (a_n - M)|$

$$\leq |(L - a_n)| + |a_n - M|$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon$$

$\Rightarrow \; |L - M| < \varepsilon$

Since ε is arbitrary, we conclude that $|L - M| = 0$

$\Rightarrow \; L - M = 0$

$\Rightarrow L = M$

Therefore, the limit of a sequence if it exists is unique.

Example 7: Prove the convergence of the following sequences by using $\varepsilon - N$ definition.

a) $\lim_{n\to\infty} \frac{1}{n} = 0$

Solution: Let $\varepsilon > 0$ be given. We need to find a positive integer N such that if $n > N$, $|a_n - L| < \varepsilon$.

Given $a_n = \frac{1}{n}$ and $L = 0$.

$|a_n - L| = \left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

Choose $N > \frac{1}{\varepsilon}$.

If $n \geq N$, then $|a_n - L| = \left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$.

Then, by definition, we have $\lim_{n\to\infty} \frac{1}{n} = 0$.

b) $\lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+1}} = 1$

Solution: Consider $\varepsilon > 0$. We need to find a positive integer N such that if $n > N$ then $|a_n - L| < \varepsilon$.

Given $a_n = \sqrt{\frac{n^2}{n^2+1}}$ and $L = 1$.

$|a_n - L| = \left|\sqrt{\frac{n^2}{n^2+1}} - 1\right| = \left|\frac{\sqrt{n^2}}{\sqrt{n^2+1}} - 1\right| = \left|\frac{n}{\sqrt{n^2+1}} - 1\right| = \left|\frac{n-\sqrt{n^2+1}}{\sqrt{n^2+1}}\right|$

Rationalizing the numerator we obtain

$\left|\frac{n^2-(n^2+1)}{(\sqrt{n^2+1})\,(n+\sqrt{n^2+1})}\right| = \left|\frac{-1}{\sqrt{n^2+1}\,(n+\sqrt{n^2+1})}\right| = \frac{1}{\sqrt{n^2+1}\,(n+\sqrt{n^2+1})}$

Observe that $(\sqrt{n^2+1})\left(n+\sqrt{n^2+1}\right) > n$ for $n \geq 1$

Consequently we get the result,

$|a_n - L| = \left|\sqrt{\frac{n^2}{n^2+1}} - 1\right| = \left|\frac{\sqrt{n^2}}{\sqrt{n^2+1}} - 1\right| = \left|\frac{n}{\sqrt{n^2+1}} - 1\right| = \left|\frac{n-\sqrt{n^2+1}}{\sqrt{n^2+1}}\right|$

$$\leq \frac{1}{\sqrt{n^2+1}\,(n+\sqrt{n^2+1})} < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Choose $N > \frac{1}{\varepsilon}$.

If $n \geq N$, we have $|a_n - L| = \left|\sqrt{\frac{n^2}{n^2+1}} - 1\right| < \varepsilon$

By definition, we have $\lim_{n\to\infty}\sqrt{\frac{n^2}{n^2+1}} = 1.$

c) $\lim_{n\to\infty}(\frac{3}{4})^n = 0$

Solution. Let $\varepsilon > 0$ be given. We need to find a positive integer N such that if $n > N$, $|a_n - L| < \varepsilon$.

Given $a_n = (\frac{3}{4})^n$ and $L = 0$.

$|a_n - L| = \left|(\frac{3}{4})^n - 0\right| = \left|(\frac{3}{4})^n\right| = (\frac{3}{4})^n \leq (\frac{3}{4})^N < \varepsilon$

Take the logarithm of both sides, $\ln\,(\frac{3}{4})^N < \ln \varepsilon$

From which we obtain $N.\ln\frac{3}{4} < \ln \varepsilon$

Since $\ln\frac{3}{4} < 0$, then dividing both sides by this value will alter the sign and we get the result $N \geq \frac{\ln \varepsilon}{\ln^3/_4}$.

Choose N to be the smallest positive integers such that $N \geq \frac{\ln \varepsilon}{\ln^3/_4}$

Then, we have $\lim_{n\to\infty}(\frac{3}{4})^n = 0.$

d) $\lim_{n\to\infty} 5 = 5$

Solution: Let $\varepsilon > 0$ be given. We need to find a positive integer N such that if $n > N$, $|a_n - L| < \varepsilon$.

Given $a_n = 5$ and $L = 5$.

$|a_n - L| = |5 - 5| = 0 < \varepsilon$ (True)

Which implies the limit value exists for all elements in the domain. Since the domain is positive integer, then choose $N \geq 1$.

Thus for any $N \geq 1$ we have $\lim_{n\to\infty} 5 = 5$

e) $\lim_{n\to\infty}\frac{1}{n^2} = 0$

Solution: Let $\varepsilon > 0$ be given. We need to find a positive integer N such that if $n > N$, $|a_n - L| < \varepsilon$.

Given $a_n = \frac{1}{n^2}$ and $L = 0$.

Now $|a_n - L| = \left|\frac{1}{n^2} - 0\right| = \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon$

Choose $N > \sqrt{\varepsilon}$

Thus for $N > \sqrt{\varepsilon}$ we have $\lim_{n\to\infty}\frac{1}{n^2} = 0$.

f) $\lim_{n\to\infty}\frac{2n-1}{n+1} = 2$

Solution: Let $\varepsilon > 0$ be given. We need to find a positive integer N such that if $n > N$ then $|a_n - L| < \varepsilon$.

Given $a_n = \frac{2n-1}{n+1}$ and $L = 2$.

Now $|a_n - L| = \left|\frac{2n-1}{n+1} - 2\right| = \left|\frac{(2n-1)-2(n+1)}{n+1}\right| = \left|\frac{-3}{n+1}\right| = \frac{3}{n+1} \leq \frac{3}{N+1} < \varepsilon$

Choose $N > \frac{3}{\varepsilon} - 1$

Therefore, for $N > \frac{3}{\varepsilon} - 1$, we have $\lim_{n\to\infty}\frac{2n-1}{n+1} = 2$.

g) $\lim_{n\to\infty}\frac{3n+1}{n} = 3$

Solution: Let $\varepsilon > 0$ be given. We need to find a positive integer N such that if $n > N$ then $|a_n - L| < \varepsilon$.

Given $a_n = \frac{3n+1}{n}$ and $L = 3$.

Now $|a_n - L| = \left|\frac{3n+1}{n} - 3\right| = \left|\frac{(3n+1)-3n}{n}\right| = \left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

Choose $N > \varepsilon$

Thus for $N > \varepsilon$, we have $\lim_{n\to\infty}\frac{3n+1}{n} = 3$.

Exercise 1.2

1. *Determine whether the sequence converges or diverges. If it converges, find the limit*

   a. $a_n = \frac{3+5n^2}{n+n^2}$

   b. $a_n = \frac{n+1}{3n-1}$

   c. $\{0,1,0,0,1,0,0,0,1, \ldots\}$

2. *Each of the following sequences $\{a_n\}$ is convergent. Thus, in case determine a value of N that is suitable for each of the following values of $\varepsilon : \varepsilon = 1$, 0.1 , 0.001, 0.0001.*

   a. $a_n = \frac{1}{n}$ b. $a_n = \frac{n}{n+1}$ c. $a_n = \frac{(-1)^{n+1}}{n}$

   d. $a_n = \frac{1}{n!}$ e. $a_n = \frac{2n}{n^3+1}$

3. *Assume that $lim_{n\to\infty} a_n = 0$, use the definition of limit to prove that $lim_{n\to\infty} {a_n}^2 = 0$.*

1.3. Convergence Properties of a Sequence

Overview

In this subtopic we are going to deal with the properties that convergent sequences will have and, verify using various examples.

Section Objectives:

On the completion of this subtopic, students will be able to:

- identify the properties that convergent sequences have.
- determine convergent sequences using the properties.

Since sequences are functions we may add, subtract, multiply and divide sequences just as we do functions.

Definition 5: Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences and c a scalar. Then

- The sum $\{a_n + b_n\}_{n=1}^{\infty}$
- Any scalar multiple $\{ca_n\}_{n=1}^{\infty}$
- The product $\{a_n b_n\}_{n=1}^{\infty}$
- The quotient $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ provided $\lim_{n\to\infty} b_n \neq 0$ are all convergent with the properties:

  i. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

ii. $\lim_{n\to\infty} c.a_n = c.\lim_{n\to\infty} a_n$

iii. $\lim_{n\to\infty} a_n.b_n = \lim_{n\to\infty} a_n . \lim_{n\to\infty} b_n$

iv. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ provided $\lim_{n\to\infty} b_n \neq 0$

v. $\lim_{n\to\infty} c = c$ where c is a constant.

vi. $\lim_{n\to\infty} {a_n}^p = (\lim_{n\to\infty} a_n)^p$ provided $p > 0$ and $a_n > 0$.

vii. $\lim_{n\to\infty} e^{a_n} = e^{\lim_{n\to\infty} a_n}$

viii. If $a_n \leq b_n$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$

Note that this statement is no longer true for strict inequalities. In other words, there are convergent sequences with $a_n < b_n$ for all n, but strict inequality is no longer true for their limits. For instance, consider $a_n = \frac{1}{n+1}$ and $b_n = \frac{1}{n}$, clearly $a_n < b_n$ but $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$.

Example 8: Find the limit of the following using the properties.

a) $\lim_{n\to\infty} \frac{2n-1}{3n+5}$

Solution: Dividing each term by the highest power of n , we get

$$\lim_{n\to\infty} \frac{2n-1}{3n+5} = \lim_{n\to\infty} \frac{2-\frac{1}{n}}{3+\frac{5}{n}}$$

$$= \frac{\lim_{n\to\infty} 2-\frac{1}{n}}{\lim_{n\to\infty} 3+\frac{5}{n}} = \frac{\lim_{n\to\infty} 2 - \lim_{n\to\infty}\frac{1}{n}}{\lim_{n\to\infty} 3 + \lim_{n\to\infty}\frac{5}{n}} = \frac{2-0}{3+0} = \frac{2}{3}$$

b) $\lim_{n\to\infty} \frac{(n+1)^n}{n^{n+1}}$

Solution: $\lim_{n\to\infty} \frac{(n+1)^n}{n^{n+1}} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n .n} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n} . \frac{1}{n}$

$$= \lim_{n\to\infty} (\frac{n+1}{n})^n . \lim_{n\to\infty} \frac{1}{n}$$

$$= \lim_{n\to\infty} (1+\frac{1}{n})^n . \lim_{n\to\infty} \frac{1}{n} = e .0 = 0$$

c) $\lim_{n\to\infty}(n - \frac{n^2}{n+1})$

Solution: $\lim_{n\to\infty}\left(n - \frac{n^2}{n+1}\right) = \lim_{n\to\infty}\left(\frac{n^2+n-n^2}{n+1}\right) = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)$

$$= \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1+\lim_{n\to\infty}\frac{1}{n}} = \frac{1}{1+0} = 1.$$

Note: There is one more simple but useful theorem that can be used to find a limit if comparable limits are known. The theorem states that if a sequence is pinched in between two convergent sequences that converge to the same limit, then the sequence in between must also converge to the same limit.

Sandwich (Squeezing) theorem 2: Given three sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ such that

- **i.** $a_n \le b_n \le c_n$ for every n , and
- **ii.** $\lim_{n\to\infty} a_n = A = \lim_{n\to\infty} c_n$

Then $\lim_{n\to\infty} b_n = A$

Proof: Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = A = \lim_{n\to\infty} c_n$, then by definition there exists positive integers N_1 and N_2 such that

$$|a_n - A| < \varepsilon \quad \text{for} \quad n \ge N_1 \qquad (1)$$

and

$$|c_n - A| < \varepsilon \quad \text{for} \quad n \ge N_2 \qquad (2)$$

Let $N = \max\{N_1 , N_2\}$, then from (1) and (2) we have

$$-\varepsilon < a_n - A < \varepsilon \qquad n \ge N \qquad (3)$$

and

$$-\varepsilon < c_n - A < \varepsilon \qquad n \ge N \qquad (4)$$

From (3) we obtain, $A - \varepsilon < a_n < A + \varepsilon$ (5)

and from (4) we get $A - \varepsilon < c_n < A + \varepsilon$ (6)

Combining (5) and (6) we get,

$$A - \varepsilon < a_n < b_n < c_n < A + \varepsilon \quad n \geq N$$

From this we have $|b_n - A| < \varepsilon$ for every $n \geq N$.

Then, by definition we have $\lim_{n\to\infty} b_n = A$.

Example 9: Find the limits of the following.

a) $\lim_{n\to\infty} \frac{1+\sin n}{n}$

Solution. We know that $-1 \leq \sin n \leq 1$

Adding 1 both sides we obtain the result,

$$0 \leq 1 + \sin n \leq 2$$

Then dividing both sides by n , we get

$$0 \leq \frac{1+\sin n}{n} \leq \frac{2}{n}$$

Taking the limit of both sides, we get $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{2}{n}$.

Thus, by sandwich theorem $\lim_{n\to\infty} \frac{1+\sin n}{n} = 0$.

b) $\lim_{n\to\infty} \frac{\sin^2 n}{n}$

Solution: We know that $0 \leq \sin^2 n \leq 1$. Then dividing both sides by n , we get the result,

$$0 \leq \frac{\sin^2 n}{n} \leq \frac{1}{n}.$$

Taking the limit of both sides we get $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty} \frac{1}{n}$.

Thus, from sandwich theorem $\lim_{n\to\infty}\frac{\sin^2 n}{n} = 0$.

c) $\lim_{n\to\infty}\sqrt{n+1}-\sqrt{n}$

Solution: Let us first rationalize the numerator

$$\sqrt{n+1}-\sqrt{n} = \frac{(\sqrt{n+1}-\sqrt{n}).(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

But we know that $\sqrt{n} \le \sqrt{n+1}+\sqrt{n} \le 2\sqrt{n+1}$

$\Longrightarrow \frac{1}{2\sqrt{n+1}} \le \frac{1}{\sqrt{n+1}+\sqrt{n}} \le \frac{1}{\sqrt{n}}$ and $\lim_{n\to\infty}\frac{1}{2\sqrt{n+1}} = 0 = \lim_{n\to\infty}\frac{1}{\sqrt{n}}$

By squeezing theorem $\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$

d) $\lim_{n\to\infty}(\frac{3n-1}{4n+1})^n$

Solution: We know that $0 \le (\frac{3n-1}{4n+1})^n \le (\frac{3}{4})^n$ and $\lim_{n\to\infty} 0 = 0 = \lim_{n\to\infty}(\frac{3}{4})^n$.

By sandwich theorem, we conclude $\lim_{n\to\infty}(\frac{3n-1}{4n+1})^n = 0$.

Techniques for computing limits of a sequence

Theorem 3: Let f be a continuous function, then $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$.

Idea of proof: By definition of continuity when $x \to x_0$ then $f(x) \to f(x_0)$.

Now $\lim_{n\to\infty} a_n = A$ means $a_n \to A$ as $n \to \infty$. Thus $f(a_n) \to f(A)$ when $n \to \infty$

That is, $\lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$.

Example 10: Find the $\lim_{n\to\infty}\sin(\frac{n\pi}{2n+1})$

Solution: $\lim_{n\to\infty}\sin(\frac{n\pi}{2n+1}) = \sin(\lim_{n\to\infty}\frac{n\pi}{2n+1}) = \sin(\lim_{n\to\infty}\frac{\pi}{2+{}^{1}/_{n}}) = \sin(\frac{\pi}{2}) = 1.$

Theorem 4 (L'Hospital's rule). Suppose $a_n = f(n)$ and $b_n = g(n)$. If $\lim_{n\to\infty}\frac{f(n)}{g(n)}$ is of the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$, then $\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{f'(n)}{g'(n)}$

Example 11: Show that

$$\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x .$$

Solution: We know that $(1+\frac{x}{n})^n = e^{n\ln((1+\frac{x}{n})}$.

Then, $\lim_{n\to\infty}(1+\frac{x}{n})^n = \lim_{n\to\infty} e^{n\ln((1+\frac{x}{n})}$

Then,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty} e^{n\ln((1+\frac{x}{n})} = e^{\lim_{n\to\infty} n\ln((1+\frac{x}{n})} = e^{\lim_{n\to\infty}\frac{\ln((1+\frac{x}{n})}{n}}$$

is $\frac{0}{0}$ form.

Then by L'Hospital's rule,

$$\lim_{n\to\infty}(1+\frac{x}{n})^n = e^x.$$

Exercise 1.3

1. *Let* $a_1 = k$, $a_2 = f(k) = f(a_1)$, $a_3 = f(a_2) = f(f(k))$, … , $a_{n+1} = f(a_n)$ *where* f *is continuous function . If* $\lim_{n\to\infty} a_n = L$ *, then show that* $f(L) = L$.
2. *Using the definition of* $\varepsilon - N$ *show that* $\lim_{n\to\infty}\frac{2n+2}{n} = 2$.
3. *Find the limit of* $\lim_{n\to\infty}\frac{5+\sin n}{n}$.

1.4. Bounded Monotone Sequences

Overview:

In this subsection we are going to deal with bounded, monotone or both bounded and monotone sequences, and comprehend the need of such in determining whether or not a particular sequence is convergent or divergent without knowing where it goes.

Section objectives

At the end of this lesson, students are expected to:

- define bounded sequence;
- define monotonic sequence;
- determine if a abounded monotonic sequence is convergent;

Definition 6: A sequence $\{a_n\}_{n=m}^{\infty}$ is bounded if there is a number M, $M > 0$ such that $|a_n| \leq M$ for every $n \geq m$ .Otherwise, we say that the sequence is unbounded.

For instance the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ and $\{1\}_{n=1}^{\infty}$ are bounded but the sequences $\{n\}_{n=1}^{\infty}$ and $\{n^2\}_{n=1}^{\infty}$ are unbounded.

Remark 5: A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be

(i). Bounded above: if there exists a number M, $M > 0$ such that $a_n \leq M$ for all $n \geq 1$.

(ii). Bounded below: if there exists a number M, $M > 0$ such that $a_n \geq M$ for all $n \geq 1$.

(iii). Bounded: if it is both above and below bounded i.e. there exists M, $M > 0$ such that $|a_n| \leq M$ for all $n \geq 1$.

Example 12: Determine whether the following sequences are bounded or unbounded.

a. $\left\{1 + \frac{2}{n}\right\}_{n=1}^{\infty}$

Solution: Here we can see that $\left|1 + \frac{2}{n}\right| \leq 3$ for every $n \geq 1$. Thus, it is bounded.

b. $\{1, 2, 3, 4, \ldots\}$

Solution: The defined formula for the sequence is $\{n\}_{n=1}^{\infty}$. Clearly, $n \geq 1$ for all $n = 1, 2, 3, \ldots$. Therefore $\{n\}_{n=1}^{\infty}$ is bounded below.

But it is not bounded above as there is no M such that the condition $n \leq M$ for every $n \geq 1$ is satisfied. Therefore, it is unbounded.

c. $\left\{e^{1/n}\right\}_{n=1}^{\infty}$

Solution: $\left|e^{1/n}\right| \leq e$ for every $n \geq 1$. It is bounded.

d. $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

Solution: $\left|\frac{1}{n}\right| \leq 1$ for every $n \geq 1$ which implies bounded.

The following theorem shows important criteria for the boundedness and divergence of sequences.

Theorem 5:

i. If $\{a_n\}_{n=1}^{\infty}$ converges, then $\{a_n\}_{n=1}^{\infty}$ is bounded.

ii. If $\{a_n\}_{n=1}^{\infty}$ is unbounded, then $\{a_n\}_{n=1}^{\infty}$ diverges.

Proof: (a). Suppose $\lim_{n\to\infty} a_n = L$, where L is a real number.

Let $\varepsilon > 0$ be given. Then by definition, there is a natural number N such that if $n > N$ then $|a_n - L| < \varepsilon$.

Choose $\varepsilon = 1$, then for $n \geq N$ we have $|a_n - L| < 1$.

Therefore if $n \geq N$, $|a_n| = |a_n - L + L|$.

By triangular inequality we have,

$|a_n - L + L| \leq |a_n - L| + |L|$

$\leq 1 + |L|$ (Since $|a_n - L| < 1$)

Let $M = \max\{ |a_1|, |a_2|, |a_3|, \ldots, |a_{N-1}|, 1 + |L| \}$

Then $|a_n| \leq M$ for all $n = 1, 2, 3, \ldots$

$\Longrightarrow \{ a_n \}_{n=1}^{\infty}$ is bounded.

(b). We proof the theorem by contradiction, that is, suppose the sequence is convergent, then from (a), the sequence is bounded. This is contradiction to the fact that the sequence is unbounded. Hence it must be divergent.

Caution: Convergence implies boundedness but boundedness does not imply convergence, i.e., bounded sequence need not be convergent.

Example 13: $\{ (-1)^n \}_{n=1}^{\infty}$ is bounded but not convergent.

Definition 7: A sequence $\{ a_n \}_{n=1}^{\infty}$ is said to be

i. Monotonically increasing if $a_n \leq a_{n+1}$ for all $n \geq 1$.
ii. Strictly increasing if $a_n < a_{n+1}$ for all $n \geq 1$.
iii. Monotonically decreasing if $a_n \geq a_{n+1}$ for all $n \geq 1$
iv. Strictly decreasing if $a_n > a_{n+1}$ for all $n \geq 1$

Remark 6:

i. An increasing (decreasing)sequence is sometimes called as non-decreasing(non-increasing) sequence.
ii. A sequence which is either increasing or decreasing is called **monotonic** sequence.

In other words, if every next member of a sequence is larger than the previous one, the sequence is growing or monotone increasing. If the next element is smaller than each previous one, the sequence is decreasing. While this condition is easy to understand, there are equivalent conditions that are often easier to check:

How to show a given sequence is monotone

Method I: First find a_{n+1} , then we say that the sequence is

- Monotone increasing if
  - $a_{n+1} \geq a_n$
  - $a_{n+1} - a_n \geq 0$
  - $\frac{a_{n+1}}{a_n} \geq 1$ if $a_n > 0$
- Monotone decreasing if
  - $a_{n+1} \leq a_n$
  - $a_{n+1} - a_n \leq 0$
  - $\frac{a_{n+1}}{a_n} \leq 1$ if $a_n > 0$

Method II. Since sequences are functions, then what is applicable for functions is also true for sequences. Therefore, using the first derivative test we can determine whether or not a given sequence is convergent or divergent.

Step I. Let $f(n) = a_n$.
Step II. Find the first derivative of the functions, i.e., $f'(x)$.
Step III. If

(i). $f'(x) > 0$ for all x or $f'(x) \geq 0$ for all x and $f'(x) = 0$ for finitely many values of x, then the sequence is increasing.

(ii). $f'(x) < 0$ for all x or $f'(x) \leq 0$ for all x and $f'(x) = 0$ for finitely many values of x, then the sequence is decreasing.

Example 14: Determine whether or not the following sequences are monotonic.

a) $\left\{\frac{1}{1+n^2}\right\}_{n=1}^{\infty}$

Solution: We have $a_n = \frac{1}{1+n^2}$. From this we obtain $a_{n+1} = \frac{1}{1+(n+1)^2} = \frac{1}{n^2+2n+2}$.

$a_n - a_{n+1} = \frac{1}{1+n^2} - \frac{1}{n^2+2n+2} = \frac{n^2+2n+2-n^2-1}{(1+n^2)(n^2+2n+2)} = \frac{2n+1}{(1+n^2)(n^2+2n+2)} > 0$ for all $n \geq 1$.

Hence, by definition, the sequence $\left\{\frac{1}{1+n^2}\right\}_{n=1}^{\infty}$ is decreasing.

b) $\left\{\frac{n}{2n+1}\right\}_{n=1}^{\infty}$

Solution: Given $a_n = \frac{n}{2n+1}$, then we obtain $a_{n+1} = \frac{n+1}{2(n+1)+1} = \frac{n+1}{2n+3}$.

$a_n - a_{n+1} = \frac{n}{2n+1} - \frac{n+1}{2n+3} = \frac{(2n^2+3n)-(2n^2+3n+1)}{(2n+1)(2n+3)} = \frac{-1}{(2n+1)(2n+3)} < 0$ for all $n \geq 1$.

Thus, by definition, the sequence $\left\{\frac{n}{2n+1}\right\}_{n=1}^{\infty}$ is increasing sequence.

c) $\left\{\frac{1}{n} - \frac{1}{n^2}\right\}_{n=1}^{\infty}$

Solution:- $a_n = \frac{1}{n} - \frac{1}{n^2}$

$\Longrightarrow a_{n+1} = \frac{1}{n+1} - \frac{1}{(n+1)^2} = \frac{1}{n+1} - \frac{1}{n^2+2n+1}$

Now $a_n - a_{n+1} = \left(\frac{1}{n} - \frac{1}{n^2}\right) - \left(\frac{1}{n+1} - \frac{1}{n^2+2n+1}\right) = \frac{n^2-n-1}{(n+1)^2 \cdot n^2} > 0$ when $n > 1$.

Therefore $\{a_n\}$ is decreasing sequence.

Bounded Monotonic sequence

Given a sequence it is not always simple to determine whether the sequence converges or diverges. For some sequences it may be suffice to know only the convergence. The following theorem will help us to know only convergence of a sequence.

Monotone convergence theorem 6: A bounded sequence $\{a_n\}_{n=1}^{\infty}$ that is either increasing or decreasing converges. That is, a bounded monotonic sequence converges.

Example 15:

a. Let $a_n = 1 + \frac{1}{1\times1!} + \frac{2}{2\times2!} + \frac{3}{3\times3!} + \ldots + \frac{1}{n\times n!}$ for $n \geq 1$. Then show that $\{a_n\}_{n=1}^{\infty}$ converges.

Solution: To show $\{a_n\}_{n=1}^{\infty}$ is convergent, we need to show if the conditions for the theorem are satisfied.

i. **Monotonocity:** Since $a_{n+1} = a_n + \frac{1}{(n+1).(n+1)!} > a_n$, then, by definition, the sequence is increasing.

ii. **Boundedness :**

$$0 < a_n = 1 + \frac{1}{1\times1!} + \frac{2}{2\times2!} + \frac{3}{3\times3!} + \ldots + \frac{1}{n\times n!}$$

$$< 1 + \frac{1}{1!} + \frac{2}{2!} + \frac{3}{3\times3!} + \ldots + \frac{1}{n!}$$

$$< e$$

Thus, the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded

Therefore, by monotone convergence theorem, the sequence convergence.

b. Let $a_1 = 1$ and define $a_{n+1} = \frac{6(1+a_n)}{7+a_n}$ for $n \geq 1$, then show that the sequence $\{a_n\}$ is convergent and find its limit.

Solution: - Before we decide the convergence or divergence of the given sequence, we have to first check whether or not the conditions for the theorem are satisfied, that is, monotonocity and boundedness.

Let $f(n) = a_{n+1}$, then the function becomes $f(x) = \frac{6(1+x)}{7+x}$ for $x \geq 1$.

Using derivative test: $f'(x) = \frac{6(1+x)'(7+x)-6(1+x).(7+x)'}{(7+x)^2} = \frac{36}{(7+x)^2} \geq 0$. This tells us the sequence is monotonically increasing.

To show boundedness:

Since $\frac{1+x}{7+x} < 1$ for all x , we obtain $6\left(\frac{1+x}{7+x}\right) \leq 6$, $\forall x \geq 0$.

Now since $a_n \geq 0$ for all n .

$f(a_n) = a_{n+1} \leq 6$ for all n . Consequently, we have $0 \leq a_{n+1} \leq 6$.

Therefore, the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded.

Thus, by monotonic convergence theorem, the sequence convergese.

To find the limit point for the sequence, assume $a_n \to L$ as $n \to \infty$, that is $\lim_{n\to\infty} a_{n+1} = L$

Now $a_{n+1} = \frac{6(1+a_n)}{7+a_n}$

Taking the limit of both sides, we obtain

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{6(1+a_n)}{7+a_n} \implies L = \frac{6(1+L)}{7+L} \implies L(7+L) = 6(1+L)$

$\implies L^2 + 7L = 6L + 6 \implies L^2 + L - 6 = 0$

From this we get $L = 2$ or $L = -3$. Since $a_n \geq 0$ for all $n \geq 1$, then $L = 2$.

c. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$ .Then show that

  i. $\{a_n\}_{n=1}^{\infty}$ is bounded sequence.
  ii. $\{a_n\}_{n=1}^{\infty}$ is monotonic sequence.
  iii. Find the limit of the sequence.

Solution:-

i. Boundedness: Using mathematical induction $a_1 = \sqrt{2} < 2$ (true).

Assume $a_n < 2$

Then we need to show $a_{n+1} < 2$.

We know that $a_n < 2$ (assumption)

Adding both sides 2, we get $2 + a_n \leq 2 + 2 = 4$

Taking the square root of both sides

$\Rightarrow \sqrt{2 + a_n} \leq \sqrt{4} = 2 \quad \Rightarrow a_n = \sqrt{2 + a_n} \leq 2$

Therefore, by the principle of mathematical induction, the sequence $\{a_n\}$ is bounded.

ii. **Monotonocity** : Again by employing mathematical induction $a_1 = \sqrt{2} \leq a_2 = \sqrt{2 + a_1} = \sqrt{2 + \sqrt{2}}$ (true).

Assume $a_n < a_{n+1}$.

We need to show $a_{n+1} < a_{n+2}$.

Add both sides 2 for the inequality on the assumption, we get $a_n + 2 < a_{n+1} + 2$.

Take the square root of both sides, we obtain $\sqrt{2 + a_n} < \sqrt{2 + a_{n+1}}$ which gives us

$a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + a_{n+1}} = a_{n+2}$.

Thus, we conclude that $a_n < a_{n+1}$ for all $n \in N$. Therefore, the sequence $\{a_n\}$ is increasing.

iii. To find its limit:

By monotone convergence theorem, we know that a bounded monotonic sequence converges, say the sequence $\{a_n\}_{n=1}^{\infty}$ converges to l .

Since $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$

$\Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2 + a_n}$

$\Rightarrow \lim_{n\to\infty} a_{n+1} = \sqrt{2 + \lim_{n\to\infty} a_n}$

$\Longrightarrow\ l = \sqrt{2+l}$

$\Longrightarrow\ l^2 = 2 + l \quad \Longrightarrow\ l^2 - l - 2 = 0 \quad \Longrightarrow\ l = 2$ or $l = -1$

Since $a_n \geq 0$ for all $n = 1,2,3, \ldots$

Thus $l = 2$.

Exercise 1.4

1. *Define the sequence* $\{a_n\}$ *by* $a_1 = 1$, $a_{n+1} = 2a_n + 2$ *for* $n \geq 1$ *.Assuming that* $\{a_n\}$ *is convergent, find its limit. Is the sequence convergent ?*
2. *Define the sequence* $\{a_n\}$ *by* $a_1 = 1$, $a_{n+1} = \frac{2}{3}a_n + \frac{2}{3{a_n}^2}$ *for* $n > 1$ *.Assuming that* $\{a_n\}$ *is convergent, find its limit.*
3. *Show that the sequence,*
   a) $a_n = \frac{n}{2n+1}$ *is increasing*
   b) $a_n = \frac{1}{n} - \frac{1}{n^2}$ *is decreasing.*
4. *Let* $a_1 = 1$ *and for* $n \geq 1$ *define* $a_{n+1} = \frac{6(1+a_n)}{1+a_n}$ *, show that* $\{a_n\}$ *is convergent and find its limit.*

Unit Summary:

1. An ordered set of numbers such as $a_1, a_2, a_3, \ldots, a_n, \ldots$ is called a sequence and usually designated by $\{a_n\}_{n=1}^{\infty}$. Each number a_k is called terms of the sequence. In particular the n^{th} term of the sequence is denoted by a_n.

2. An infinite sequence (sequence) is a function, say f, whose domain is the set of all integers greater than or equal to some integer m (usually 0 or 1). If f is an integer greater than or equal to some integer m and $f(n) = a_n$, then we express the sequence by writing its range in any of the following ways:

$f(m), f(m+1), f(m+2), \ldots$; $a_m, a_{m+1}, a_{m+2}, \ldots$; $\{f(n) : n \geq m\}$; $\{f(n)\}_{n=m}^{\infty}$ or $\{a_n\}_{n=m}^{\infty}$

3. a). Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. A number L is the limit of $\{a_n\}_{n=1}^{\infty}$ or the sequence $\{a_n\}_{n=1}^{\infty}$ converges to L if for every $\varepsilon > 0$, there exists a positive integer $\mathbb{N}$ such that if $n > \mathbb{N}$, then $|a_n - L| < \varepsilon$. In this case we write $\lim_{n\to\infty} a_n = L$ or $a_n \to L$ as $n \to \infty$.

If such a number L does not exist, the sequence has no limit or diverges.

b). A sequence $\{a_n\}_{n=1}^{\infty}$ is said to diverge to

(i). ∞ if for every positive number M, there is an positive integer N such that if $n > N$, then $a_n > M$, and write it as $\lim_{n\to\infty} a_n = \infty$.

(ii). $-\infty$ if for every positive number M, there is a positive integer N such that if $n > N$, then $a_n < M$. We write it as $\lim_{n\to\infty} a_n = -\infty$.

4. If $\lim_{n\to\infty} a_n = L$ exists, we say that $\{a_n\}_{n=1}^{\infty}$ converges (converges to L). If such a number L does not exist, we say that the sequence $\{a_n\}_{n=1}^{\infty}$ diverges or that $\lim_{n\to\infty} a_n$ does not exist.

5. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are convergent sequences and c a scalar. Then

the sum $\{a_n + b_n\}_{n=1}^{\infty}$; Any scalar multiple $\{ca_n\}_{n=1}^{\infty}$; The product $\{a_nb_n\}_{n=1}^{\infty}$

and the quotient $\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty}$ provided $\lim_{n\to\infty} b_n \neq 0$ are all convergent.

6. Sandwich (Squeezing) theorem: Given three sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ such that

i. $a_n \leq b_n \leq c_n$ for every n , and
ii. $\lim_{n\to\infty} a_n = A = \lim_{n\to\infty} c_n$.

Then $\lim_{n\to\infty} b_n = A$

7. A sequence $\{a_n\}_{n=m}^{\infty}$ is bounded if there is a number M , $M > 0$ such that $|a_n| \leq M$ for every $n \geq m$ .Otherwise, we say that the sequence is unbounded.

8. A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be

(i). Bounded above: if there exists a number M , $M > 0$ such that $a_n \leq M$ for all $n \geq 1$.

(ii). Bounded below: if there exists a number M , $M > 0$ such that $a_n \geq M$ for all $n \geq 1$.

(iii). Bounded: if it is both above and below bounded i.e. there exists M , $M > 0$ such that $|a_n| \leq M$ for all $n \geq 1$.

9. Convergence implies boundedness but boundedness does not imply convergence, i.e., bounded sequence need not be convergent.

Example: $\{ (-1)^n \}_{n=1}^{\infty}$ is bounded but not convergent.

12) Let $\{ a_n \}_{n=m}^{\infty}$ be given sequence. Then $\{ a_n \}_{n=m}^{\infty}$ is said to be

i. Monotonic increasing if $a_n \leq a_{n+1}$ for all $n \geq 1$.
ii. Strictly increasing if $a_n < a_{n+1}$ for all $n \geq 1$.
iii. Monotonic decreasing if $a_n \geq a_{n+1}$ for all $n \geq 1$.
iv. Strictly decreasing if $a_n > a_{n+1}$ for all $n \geq 1$.

14) **Monotone convergence theorem:-** A **bounded** sequence $\{a_n\}_{n=m}^{\infty}$ that is either **increasing** or **decreasing** converges. That is, a bounded monotonic sequence converges.

Miscellaneous Exercises

1. *Find a formula for the general term* a_n *of the sequence, assuming that the pattern of the first few terms continues.*

a. $\left\{1, \frac{-2}{3}, \frac{4}{9}, \frac{-8}{27}, \dots\right\}$ b. $\{5,10,15,20, \dots\}$ c. $\{7,2,7,2,7,2, \dots\}$

2. *Using the definition of sequence* $(\varepsilon - N)$, *verify the following*

a. $\lim_{n\to\infty} \frac{2n-1}{n} = 2$ b. $\lim_{n\to\infty} \frac{1}{n^2} = 0$ c. $\lim_{n\to\infty} c = c$, *where c is a constant.*

3. *Determine whether the sequence converges or diverges. If it converges, find the limit.*

a. $a_n = \frac{n}{2n+1}$ b. $a_n = \frac{n+1}{3n+1}$ c. $\{3,1,3,3,1,3,3,3,1, \dots\}$

d. $a_n = \frac{(n+1)^n}{n^{n+1}}$ e. $a_n = \frac{2n^2+1}{2n+1}$ f. $a_n = n - \frac{n^2}{n+1}$

4. a. *If* $\lim_{n\to\infty} a_n = L$, *then what is the limit of* $\lim_{n\to\infty} a_{n+1}$? *why ?*

b. *Using (a), find the limit of the sequence* $\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots\right\}$?.

5. *Prove that the sequence* $\{a_n\}_{n=1}^{\infty}$ *converges to zero if and only if the sequence* $\{|a_n|\}_{n=1}^{\infty}$ *converges to zero.*

6. *Prove that if the sequence* $\{a_n\}_{n=1}^{\infty}$ *converges to a number* A , *then the sequence* $\{|a_n|\}_{n=1}^{\infty}$ *converges to a number* $|A|$. *Is the converse true? Give an example of a divergent sequence* $\{a_n\}_{n=1}^{\infty}$ *such that the sequence* $\{|a_n|\}_{n=1}^{\infty}$ *is convergent.*

7. *Prove that the following are null sequences*

i. $\left\{\frac{n!}{n^n}\right\}_{n=1}^{\infty}$ ii. $\left\{\frac{\sin n}{n}\right\}_{n=1}^{\infty}$ iii. $\left\{\frac{c^n}{n!}\right\}_{n=1}^{\infty}$ *where c is any fixed real number.*

8. *Determine whether the sequence is increasing, decreasing or not monotonic. Is the sequence bounded ?*

a. $a_n = \frac{1}{2n+3}$ *b.* $a_n = \frac{2n-3}{3n+4}$ *c.* $a_n = cos\left(\frac{n\pi}{2}\right)$ *d.* $a_n = n + \frac{1}{n}$

9. *Suppose that* $\{a_n\}$ *is decreasing sequence and all its terms lie between the numbers 6 and 9. Does the sequence have a limit? If so, what can you say about the value of the limit?*

10. *Show that the sequence defined by* $a_1 = 1$, $a_{n+1} = 3 - \frac{1}{a_n}$ *is increasing and* $a_n < 3$ *for all* n. *Deduce that* $\{a_n\}$ *is convergent and find its limit .*

11. *Show that the sequence defined by* $a_1 = 2$, $a_{n+1} = 3 - \frac{1}{a_n}$ *satisfies* $0 < a_n \leq 2$ *and is decreasing. Deduce that the sequence is convergent and find its limit.*

12. *A sequence is defined recursively by* $a_1 = 1$, $a_{n+1} = 1 + \frac{1}{1+a_n}$, *then*

a. *Find the first five terms of the sequence* $\{a_n\}$.
b. *Show that* $\{a_n\}$ *is convergent and deduce that* $lim_{n\to\infty} a_n = \sqrt{2}$.
c. *Find* $1 + \frac{1}{2+\frac{1}{2+\cdots}}$

14. *Let* $\{a_n\}_{n=1}^{\infty}$ *be the sequence* $\sqrt{2}$, $(\sqrt{2})^{\sqrt{2}}$, $(\sqrt{2})^{(\sqrt{2})^{\sqrt{2}}}$, ... *where in general*

$a_{n+1} = (\sqrt{2})^{a_n}$, *then*

a. *Show that* $\{a_n\}_{n=1}^{\infty}$ *is bounded sequence.*
b. *Show that* $\{a_n\}_{n=1}^{\infty}$ *is increasing sequence.*
c. *Show that* $\{a_n\}_{n=1}^{\infty}$ *converges and find the limit.*

15. *Suppose that* $0 < a_n \leq a_{n+1} < M$ *for each natural number* n . *Then, prove that*

i. $\{a_n\}_{n=1}^{\infty}$ *converges*
ii. $\{-a_n\}_{n=1}^{\infty}$ *converges*
iii. $\{{a_n}^k\}_{n=1}^{\infty}$ *converges for each natural number k*

16. Define the sequence $\{a_n\}$ *by* $a_1 = 1$ *,* $a_{n+1} = \frac{4a_n+2}{a_n+3}$ *for* $n \geq 1$ *.Assuming that* $\{a_n\}$ *is convergent, find its limit.*

17. Define the sequence $\{a_n\}$ *by* $a_1 = 1$ *,* $a_{n+1} = \frac{2}{3}a_n + \frac{2}{3a_n{}^2}$ *for* $n > 1$ *.Assuming that* $\{a_n\}$ *is convergent, find its limit.*

18. If $a_n = \frac{10^n}{n!}$ *, then find a positive integer* N *such that* $a_{n+1} < a_n$ *when* $n \geq N$*.*

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Chapter two

Infinite series

Introduction

This section considers a problem of adding together the terms of a sequence. Of course, this is a problem only if more than a finite number of terms of a sequence are non zero. In this case we must decide, what it meant to add together an infinite number of nonzero numbers.

The question of dividing a line segment into infinitesimal parts has stimulated the imaginations of philosophers for a very long time. In a corruption of paradox introduced by Zeno of Elea (in the fifth century B.C.) a dimension less frog sits on the end of a one dimensional log of unit length. The frog jumps halfway, and then halfway and infinitum. The question is whether the frog ever reaches the other end. Mathematically, an unending sum,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}+\cdots$$

is suggested. "Common sense" tells us the sum must approach one even though that value is never attained. We can form partial sums,

$$S_1=\frac{1}{2}, \quad S_2=\frac{1}{2}+\frac{1}{4}, \quad S_3=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}, \ldots, S_n=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}$$

and then examine the limit.

In this chapter, consideration of such sums launches us on the road to the theory of infinite series; which are sums that involve infinitely many terms of the sequence. Infinite series play a fundamental role in both mathematics and science- they are used, for example, to approximate trigonometric functions and logarithmic functions, to solve a differential equations, to evaluate difficult integrals, to create new functions, and to construct mathematical models of physical laws. Since it is impossible to add up infinitely many numbers directly, our first goal will be to define exactly what we mean by the sum of an infinite series. However, unlike finite sums, it turns out that not all infinite series actually have a sum, so we will need to develop tool for determining which infinite series have sum and which do not. Once the basic ideas have been developed we will begin to apply our work; we will show how infinite series are used to evaluate such quantities as $\sin 17^{\circ}$ and $\ln 5$, how they are used to create functions, and finally, and how they are used to model physical laws.

Unit Objectives:

On the completion of this unit, you should be able to:

- Understand the definition of an infinite series;
- Understand the idea of partial sum in series
- Define convergence or divergence of a series;
- determine whether a given series is convergent or divergent;
- Distinguish the different types of series ;
- Define alternating series;
- Determine the convergence of alternating series;
- Understand the different types convergence tests of a series;
- Understand the idea of absolute and conditional convergence;
- Understand the different types of convergent tests for absolute convergent series;
- realize the need for the use of non negative sequence in convergent tests;

2.1. Definition and examples of infinite series

Overview:

So far we have introduced a sequence, and in this subsection, we are going to add these elements of the sequence so as to obtain an infinite series. Moreover, the partial sums of a sequence and the partial sums of a series which helps us in determining the convergence of a series will be dealt on.

Section Objectives:

On the completion of this subtopic, students will be able to:

- define an infinite series;
- determine the partial sums of a sequence; partial sums of a series;

Definition 1: Let $\{ a_n \}_{n=1}^{\infty}$ be a sequence of real numbers, then the expression $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ which is denoted by $\sum_{i=1}^{\infty} a_i$, that is, $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is called an infinite series.

Remark 1: Consider the series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$, then

i. a_n is called the n^{th} - term of the series and $a_n = S_n - S_{n-1}$.

ii. Let $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$, then S_n is called the n^{th} - partial sum of the series.

iii. $\{ S_n \}_{n=1}^{\infty}$ where S_n is the n^{th} – partial sum is called the sequence of partial sums.

Definition 2: A series is the sum of the terms of the sequence which is infinite in number i.e. $a_1 + a_2 + a_3 + \ldots + a_n + \cdots$ is called an infinite series and is usually denoted by $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$ or s_{∞}.

Sums of an infinite series

To define the sum of an infinite series, we require the definition of partial sums

Definition 3: Let $\sum_{n=1}^{\infty} a_n$ be an infinite series. Let $s_n = a_1 + a_2 + a_3 + \ldots + a_n$,then $s_n = \sum_{k=1}^{n} a_k$ is called the n^{th} – partial sum of the series and $\{ S_n \}_{n=1}^{\infty}$ where S_n is the n^{th} – partial sum is called the sequence of partial sums of the series.

Example 1:

a. For each positive integer n, assume $a_n = 1$, then find

i. The series?

ii. The n^{th} – partial sum of the series?

iii. The sequence of partial sums?

Solution:The general term of the sequence $\{ a_n \}_{n=1}^{\infty}$ is given by $a_n = 1$, and from this we obtain the first few terms of the sequence as : $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, ...

i. The series is given by $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \cdots$

$$= 1 + 1 + 1 + \cdots \quad \text{or} \quad = \sum_{n=1}^{\infty} 1$$

ii. The n^{th} – partial sum of the series is given by

$$S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \dots + a_n$$
$$= \underbrace{1 + 1 + 1 + \cdots + 1}_{n-times} = n \times 1 = n$$

Hence $s_n = n$

iii. The sequence of partial sums is given by $\{ S_n \}_{n=1}^{\infty} = \{ n \}_{n=1}^{\infty} = \{1,2,3,4, \dots\}$ or $\{ S_n \}_{n=1}^{\infty} = \{ s_1, s_2, s_3, s_4, \dots\}$ where $s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3$, $s_4 = a_1 + a_2 + a_3 + a_4$

Therefore, $\{ S_n \}_{n=1}^{\infty} = \{ s_1, s_2, s_3, s_4, \dots\} = \{1,2,3,4, \dots\}$

b. Let $a_n = \frac{1}{n(n+1)}$, then find

i. The series?

ii. The n^{th} – partial sum of the series?

iii. The sequence of partial sums?

Solution: given the n^{th} –term of the sequence $a_n = \frac{1}{n(n+1)}$, then from this we obtain $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{6}$, $a_3 = \frac{1}{12}$, $a_4 = \frac{1}{20}$, ...

i. The series is given by

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \cdots$$
$$= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots \quad \text{Or} \quad \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

ii. The n^{th} partial sum of the series is given by

$$S_n = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n$$
$$= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \cdots + \frac{1}{n(n+1)} \quad \text{Or} \quad s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)}$$

iii. The sequence of partial sums is given by $\{ S_n \}_{n=1}^{\infty} = \{ s_1, s_2, s_3, s_4, \dots\}$

Where $s_1 = a_1 = \frac{1}{2}$, $s_2 = a_1 + a_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$,

$s_3 = a_1 + a_2 + a_3 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$, $s_4 = a_1 + a_2 + a_3 + a_4 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{3}{10}$

Therefore $\{ S_n \}_{n=1}^{\infty} = \{ s_1, s_2, s_3, s_4, \dots\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{3}{10}, \dots\}$

Note:- $\{ S_n \}_{n=1}^{\infty} \neq \{ a_n \}_{n=1}^{\infty}$, the first is to mean the sequence of partial sums of the series while the later is a sequence.

Exercise2.1:

1. *For each positive integer n, assume $a_n = 2$, then*

a. *Find the series*

b. *The n^{th} partial sum of the series*

c. *The sequence of partial sums*

2. *If the n^{th} partial sum of the series is given by $S_n = \frac{n-1}{n+1}$, then find the*

a. *Third term of the sequence i.e. a_3*

b. *n^{th} Term of the sequence i.e. a_n*

c. $\sum_{n=1}^{\infty} a_n$

3. *If the n^{th} partial sum of the series is given by $S_n = n^2 + 3n$, then find the*

a. *Third term* **b**. *fifth term* **c**. *n^{th} term of the sequence*

4. *Given the series, then find the n^{th} partial sum of the series i.e. S_n and the sum of the series*

a. $\sum_{n=0}^{\infty} \frac{1}{n^2+3n+2}$ *b.* $\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)}$ *c.* $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ *d.* $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$

e. $a_n = \frac{2}{n+1} - \frac{2}{n+2}$

2.2. Convergence and divergence of a series

Overview:

In this subsection, we are going to deal with the convergence and divergence of a series. In line with, we discuss about geometric series and their convergence.

Section Objective

After completing this subtopic, students will be able to:

- define the convergence and divergence of series;
- determine convergent and divergent series;
- determine geometric series;

Definition 4:

- **Convergent series**. An infinite series $\sum_{n=1}^{\infty} a_n$ with sequence of partial sums $\{ S_n \}_{n=1}^{\infty}$ is said to be convergent if and only if the sequence of partial sums $\{ S_n \}_{n=1}^{\infty}$ converges, i.e., if $\lim_{n\to\infty} s_n$ exists, then we say that the series $\sum_{n=1}^{\infty} a_n$ is a convergent series and we write it as $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$.
- **Divergent series**. A series $\sum_{n=1}^{\infty} a_n$ is said to be divergent if it is not convergent, i.e., the series $\sum_{n=1}^{\infty} a_n$ is divergent if and only if the sequence of partial sums $\{ S_n \}_{n=1}^{\infty}$ is divergent.

Theorem 1: If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Proof: Suppose that the series $\sum_{n=1}^{\infty} a_n$ converges to L, then we need to show that $\lim_{n\to\infty} a_n = 0$.

The fact that $a_n = s_n - s_{n-1}$ and taking limit of both sides we obtain,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = L - L = 0 .$$

Example 2:

Prove that the following series are convergent and find the limit of the series

a. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Solution: Given $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

Now $s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

Then,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges and it converges to the sum 1.

b. $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

Solution: We have $a_n = \frac{3}{n(n+3)} = \frac{1}{n} - \frac{1}{n+3}$.

Now $s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+3}\right)$

$= \left(1 - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n+2}\right) + \left(\frac{1}{n} - \frac{1}{n+3}\right)$

$= \frac{11}{6} - \frac{1}{n+3}$

Then,

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{11}{6} - \frac{1}{n+3}\right) = \frac{11}{6}$$

Therefore the series $\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$ converges and it converges to the sum $\frac{11}{6}$.

c. $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$

Solution: It is given that $a_n = \frac{2}{n^2+4n+3} = \frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$

$s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left(\frac{1}{k+1} - \frac{1}{k+3}\right)$

$= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2}\right) +$
$+ \left(\frac{1}{n+1} - \frac{1}{n+3}\right)$.

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{n+3} = \frac{5}{6} - \frac{1}{n+3}$

Since

$$\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} = \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{5}{6} - \frac{1}{n+3}\right) = \frac{5}{6}$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$ converges and it converges to the sum $\frac{5}{6}$.

Divergence test (The n^{th} − term test). If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Remark 2: If $\sum_{n=1}^{\infty} a_n$ is convergent series, then $\lim_{n\to\infty} a_n = 0$. The converse need not be always true, i.e., $\lim_{n\to\infty} a_n = 0$ does not imply the series $\sum_{n=1}^{\infty} a_n$ converges. If

$\lim_{n\to\infty} a_n = 0$, then we cannot drawn any conclusion about the convergence or divergence of the series $\sum_{n=1}^{\infty} a_n$.

Example 3: Consider the series $\sum_{n=1}^{\infty}\frac{1}{n}$. Here we have $a_n = \frac{1}{n}$ and also $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$, however, the series $\sum_{n=1}^{\infty}\frac{1}{n}$ is a divergent.

Examples: Use divergence test to test the divergence of the following series:

a. $\sum_{n=1}^{\infty}(-1)^n$

Solution: given $a_n = (-1)^n$, then find its limit, i.e., $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(-1)^n$ does not exist; by the n^{th} term divergence test (divergence test) $\sum_{n=1}^{\infty}(-1)^n$ diverges.

b. $\sum_{n=1}^{\infty}\cos(\frac{2}{n})$

Solution: given $a_n = \cos(\frac{2}{n})$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\cos(\frac{2}{n}) = \cos(\lim_{n\to\infty}\frac{2}{n}) = \cos(0) = 1 \neq 0$.

By the n^{th} term divergence test $\sum_{n=1}^{\infty}\cos(\frac{2}{n})$ diverges.

c. $\sum_{n=1}^{\infty}(1-\frac{1}{2n})^n$

Solution: given $a_n = (1-\frac{1}{2n})^n$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(1-\frac{1}{2n})^n = (\lim_{m\to\infty}(1+\frac{1}{m})^m)^{-\frac{1}{2}} = e^{-\frac{1}{2}} \neq 0$.

By the n^{th} term divergence test, the series $\sum_{n=1}^{\infty}(1-\frac{1}{2n})^n$ diverges.

d. $\sum_{n=1}^{\infty}\frac{3n^2-n+4}{2n^2+1}$

Solution: given $a_n = \frac{3n^2-n+4}{2n^2+1}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3n^2-n+4}{2n^2+1} = \frac{3}{2} \neq 0$.

Thus, by the n^{th} term divergence test, the series $\sum_{n=1}^{\infty}\frac{3n^2-n+4}{2n^2+1}$ diverges.

Geometric Series

When a sequence has a constant ratio between successive terms it is called a geometric sequence and the constant is called the common ratio denoted by r.

Theorem 2: Suppose that a, $a \neq 0$ and r are real numbers, then the geometric series is given by:

$$a_1 + a_2 r + a_3 r^2 + a_4 r^3 + \cdots + \cdots = \sum_{n=1}^{\infty} a_n r^{n-1}$$

and,

a. $\sum_{n=1}^{\infty} ar^n$ converges if $|r| < 1$ and $S_\infty = \frac{1}{1-r}$

b. $\sum_{n=1}^{\infty} ar^n$ diverges if $|r| \geq 1$

Example 4: Test the convergence and divergence of the following geometric series

a. $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots$

Solution: the series is a geometric series with $a_1 = 1$ and $r = -\frac{2}{3}$ and since $|r| = \frac{2}{3} < 1$, then the geometric series converges and its sum is given by: $S_\infty = \frac{1}{1-r} = \frac{1}{1-(\frac{-2}{3})} = \frac{3}{5}$.

b. $\sum_{n=0}^{\infty} \left(\frac{4}{3}\right)^n$

Solution: since the common ratio $|r| = \frac{4}{3} > 1$, then the geometric series diverges.

Definition 5: (Harmonic Series) The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is called harmonic series and it diverges.

Example 5: Change the following into its equivalent fraction using geometric series

a. $1 + 0.4 + 0.16 + 0.064 + \cdots$

Solution: $1 + 0.4 + 0.16 + 0.064 + \cdots$

$$= 1 + \frac{4}{10} + (\frac{4}{10})^2 + (\frac{4}{10})^3 + (\frac{4}{10})^4 + \cdots$$

The series is a geometric series with $r = \frac{4}{10} < 1$, hence converges and it converges to

$S_\infty = \sum_{n=1}^{\infty} a_n = \frac{1}{1-r} = \frac{1}{1-\frac{2}{5}} = \frac{5}{3}$.

b. $0.1\dot{2}\dot{3}$

Solution: $0.1\dot{2}\dot{3} = 0.1 + 0.0\dot{2}\dot{3} = \frac{1}{10} + \frac{23}{1000} + \frac{23}{100{,}000} + \frac{23}{10{,}000{,}000} + \cdots$

$$= \frac{1}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \frac{23}{10^7} + \cdots = \frac{1}{10} + \frac{23}{10^3}(1 + \frac{1}{10^2} + \frac{1}{10^4} + \frac{1}{10^6})$$

$$= 0.1 + \frac{23}{10^3}\left(\frac{1}{1-\frac{1}{100}}\right) = 0.1 + \frac{23}{990} .$$

Exercise 2.2

1. *Given the series* $\sum_{n=2}^{\infty}\frac{1}{n(n-1)}$ *, then*
   (i). *Find* S_5 , S_6 *and* S_n?
   (ii). *If it is convergent, find its sum ?*
2. *Find the sums of the following series*

   a. $\sum_{n=0}^{\infty}\frac{1}{n^2+3n+2}$ **b.** $\sum_{n=2}^{\infty}\frac{2}{n^2-1}$ **c.** $\sum_{n=1}^{\infty}\frac{2}{n^2+4n+1}$ **d.** $\sum_{n=1}^{\infty} ln\frac{n}{n+1}$
3. *If the* n^{th} *partial sum of the series* $\sum_{n=1}^{\infty} a_n$ *is* $s_n = \frac{n-1}{n+1}$, *then find* a_n *and* $\sum_{n=1}^{\infty} a_n$?
4. *Prove that* $\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\cdots=\sum_{n=1}^{\infty}\frac{1}{(2n-1)(2n+1)}$ *converges and find its sum.*

2.3. Properties of convergent series

Overview

In this section, we are going to discuss about convergent series and their properties in detail. Also we are going to determine whether or not a particular series is convergent or divergent by using these properties.

Section Objective

On the completion of this subtopic, students should be able to:

- explain the properties of convergent series;
- to use the properties of convergent series so as to determine if a particular series is convergent;

Theorem 3: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series and c is a real number, then

i. $\sum_{n=1}^{\infty}(a_n \pm b_n)$ converges and $\sum_{n=1}^{\infty}(a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$

ii. $\sum_{n=1}^{\infty} ca_n$ converges and $\sum_{n=1}^{\infty} c.a_n = c.\sum_{n=1}^{\infty} a_n$

Caution: The product series $\sum_{n=1}^{\infty} a_n b_n$ may or may not be convergent.

Example 6:

a. Find $\sum_{n=1}^{\infty}(\frac{1}{2^n} + \frac{1}{3^n})$

Solution: $\sum_{n=1}^{\infty}(\frac{1}{2^n} + \frac{1}{3^n}) = \sum_{n=1}^{\infty}\frac{1}{2^n} + \sum_{n=1}^{\infty}\frac{1}{3^n} = \frac{\frac{1}{2}}{1-\frac{1}{2}} + \frac{\frac{1}{3}}{1-\frac{1}{3}} = 1 + \frac{1}{2} = \frac{3}{2}$

b. Find $\sum_{n=1}^{\infty}\frac{1+2^n}{3^n}$

Solution: $\sum_{n=1}^{\infty}\left(\frac{1+2^n}{3^n}\right) = \sum_{n=1}^{\infty}\left[\frac{1}{3^n} + \left(\frac{2}{3}\right)^n\right] = \sum_{n=1}^{\infty}\frac{1}{3^n} + \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$

$$= \frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{2}{3}}{1-\frac{2}{3}} = \frac{1}{2} + 2 = \frac{5}{2}$$

c. $\sum_{n=1}^{\infty}(\frac{4}{2^n} - \frac{2}{n(n+1)})$

Solution: $\sum_{n=1}^{\infty}\left(\frac{4}{2^n} - \frac{2}{n(n+1)}\right) = \sum_{n=1}^{\infty}\frac{4}{2^n} - \sum_{n=1}^{\infty}\frac{2}{n(n+1)} = 4.\sum_{n=1}^{\infty}\frac{1}{2^n} - 2.\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$

$$= 4(1) - 2(1) = 2$$

d. $\sum_{n=1}^{\infty} 2^{2n}.6^{1-n}$

Solution:

$\sum_{n=1}^{\infty} 2^{2n}.6^{1-n} = \sum_{n=1}^{\infty} 2^{2n}.6.6^{-n} = \sum_{n=1}^{\infty}\frac{2^{2n}}{6^n}.6 = 6.\sum_{n=1}^{\infty}\frac{4^n}{6^n} = 6.\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$

$$= 6\left(\frac{\frac{2}{3}}{1-\frac{2}{3}}\right) = 6(2) = 12$$

Remark 3:

1. The series $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ is called **telescoping series** because when we write the partial sums, all but the first and the last terms cancel.

2. A finite number of terms do not affect the convergence or divergence of a series.

i.e. if $\sum_{n=N+1}^{\infty} a_n$ converges, then the full series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$

Example 7: Given $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \ln 2$, then find the sum of $\sum_{n=4}^{\infty}\frac{(-1)^{n+1}}{n}$?

Solution: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \sum_{n=1}^{3}\frac{(-1)^{n+1}}{n} + \sum_{n=4}^{\infty}\frac{(-1)^{n+1}}{n}$ and $\ln 2 = (1 - \frac{1}{2} + \frac{1}{3}) + \sum_{n=4}^{\infty}\frac{(-1)^{n+1}}{n}$. From which we get $\sum_{n=4}^{\infty}\frac{(-1)^{n+1}}{n} = \ln 2 - \frac{5}{6}$.

Exercise 2.3

1. *If* $\sum_{n=1}^{\infty}\frac{1+2^n}{3^n} = {}^5/_2$, *then find* $\sum_{n=3}^{\infty}\frac{1+2^n}{3^n}$?
2. *If* $n + n^2 + n^3 + \cdots = 2n$, *then what is the value of* n ?
3. *If* $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$, *then find* $\sum_{n=4}^{\infty}\frac{1}{n^2}$?

2.4. Convergence test for positive term series (Integral test, comparison, ratio and root test)

Given a series it is not always simple to determine whether the series converges or diverges. For some series it may be suffice to know only the convergence. The following convergence tests will help us to know only convergence of a theorem.

Definition 6:

- **Non negative series:** If $a_n \geq 0$ (non negative) for every positive integer n, then the series $\sum_{n=1}^{\infty} a_n$ is called a non negative series.
- **Positive Series:** If $a_n > 0$ (positive) for every positive integer n, then the series $\sum_{n=1}^{\infty} a_n$ is called a positive term series or positive series.

Remark 4: Consider a positive series $\sum_{n=1}^{\infty} a_n$ i.e. $a_n > 0$ for all n, the series of partial sums of $\sum_{n=1}^{\infty} a_n$ increases and hence the series $\sum_{n=1}^{\infty} a_n$ converges if and only if its sequence of partial sums is bounded.

Example 8: The following are some positive series

a. $1 + 2 + 4 + 8 + \cdots$

b. $\sum_{n=1}^{\infty}(\frac{2}{3})^n$

Tests of convergence for positive series

Q1. Why we need convergence tests?

Solution: For most series the exact sum is difficult or impossible to find. It may be suffice to know at least that the series converges.

Q2. Why, then, non negative or positive series is considered?

Solution:

- Because the study of their convergence is comparatively simple and can be used in determination of convergence of more general series whose terms are not necessarily positive.
- It is easy to see that a series of positive terms diverge if and only if its sum is $+\infty$.

Tests of convergence

There are two types of convergence tests: one that compares a non negative series with an improper integral and one that compares a given non- negative (positive) series with another series.

2.4.1. The integral test

This test compares a non- negative series with an improper integral.

Integral test theorem 4: Let $\{a_n\}_{n=1}^{\infty}$ be a non- negative sequence and f be a continuous decreasing function defined on $[1, \infty)$ such that $f(n) = a_n$ for all $n \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)dx$ converges.

Remark 5:

1. When we use the integral test, it is not necessary to start the series or the improper integral at $n = 1$.

   Example 9: in testing the series $\sum_{n=4}^{\infty} \frac{1}{(n-3)^2}$, we use the improper integral $\int_4^{\infty} \frac{1}{(x-3)^2} dx$.
2. It is not necessary that f be always decreasing; what is important is that f be ultimately decreasing for x larger than some number n.
3. Since the initial few terms of the series do not affect its convergence, we may sometimes define the integral test on the interval different from $[1, \infty)$.
4. The integral test is most effective when the function f to be used is easily integrated

Example 10: Determine whether or not the following series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Solution: given $a_n = \frac{\ln n}{n}$, from which we get $\{a_n\}_{n=1}^{\infty} = \{\frac{\ln n}{n}\}_{n=1}^{\infty}$ is a non negative sequence.

The function $f(x) = \frac{\ln x}{x}$ is non negative and continuous for $x \geq 1$ for logarithm function is continuous.

To check if the function is decreasing we need to compute its derivative, that is,

$f'(x) = (\frac{\ln x}{x})' = \frac{(\ln x)'.x - \ln x}{x^2} = \frac{\frac{1}{x}.x - \ln x}{x^2} = \frac{1-\ln x}{x^2}$. Thus $f'(x) < 0$ when $\ln x > 1$, i.e., when $x > e$. Therefore, f is decreasing when $x > e$.

Applying the integral test:

$$\int_1^\infty f(x)dx = \int_1^\infty \frac{\ln x}{x} dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x} dx$$

$$= \lim_{t\to\infty} \frac{(\ln x)^2}{2}\Big|_1^t = \lim_{t\to\infty} \frac{1}{2}[(\ln t)^2 - (\ln 1)^2] = \frac{1}{2}.\lim_{t\to\infty}(\ln t)^2 = \infty$$

The improper integral is divergent implies the series $\sum_{n=1}^\infty \frac{\ln n}{n}$ also diverges by the integral test.

b. $\sum_{n=2}^\infty \frac{1}{n \ln n}$

Solution: let $f(x) = \frac{1}{x \ln x}$ for $x \geq 2$. Clearly f is continuous and decreasing on $[2, \infty)$.

To check if f is decreasing we have

$f'(x) = (\frac{1}{x \ln x})' = \frac{(1)'.\ x\ln x - (x \ln x)'}{(x \ln x)^2} = \frac{-(1+\ln x)}{(x \ln x)^2} < 0$ for $x \geq 2$ which implies f is decreasing.

Now $\int_2^\infty f(x)dx = \int_2^\infty \frac{1}{x.\ln x} dx = \lim_{t\to\infty} \int_2^t \frac{1}{x.\ln x} dx$

$$= \lim_{t\to\infty}(\ln(\ln x))|_2^t = \lim_{t\to\infty}[\ln(\ln t) - \ln(\ln 2)] = \infty$$

Hence, by the integral test $\sum_{n=2}^\infty \frac{1}{n \ln n}$ diverges.

The P-Series

Definition 7 (**The p-series):** The series $\sum_{n=1}^\infty \frac{1}{n^p}$ where p a real number is called a pseries.

Theorem 5(**p-series test**) The p- series $\sum_{n=1}^\infty \frac{1}{n^p}$

- **i.** Converges if $p > 1$
- **ii.** Diverges if $p \leq 1$.

Proof: we use the integral test to prove the p-series test.

Case I. When $p \leq 0$

The term $\frac{1}{n^p}$ does not tend to 0 as $n \to \infty$. Thus by divergence test, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges.

Case II. When $p > 0$

If $p = 1$, the series is harmonic series; consequently, the series is divergent.

Assume $p > 0$ and $p \neq 1$.

Let $f(x) = \frac{1}{x^p}$ for $x \geq 1$, then f is decreasing and continuous on $[1, \infty)$.

Thus $\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x^p}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}dx = \lim_{t\to\infty} \frac{1}{1-p} . x^{1-p}|_1^t$

$= \frac{1}{1-p}\lim_{t\to\infty}(t^{1-p} - 1) = \frac{1}{1-p}\lim_{t\to\infty}(\frac{1}{t^{p-1}} - 1) = \frac{1}{1-p}\lim_{t\to\infty} \frac{1}{t^{p-1}} - \frac{1}{1-p}$

If $p > 1$, $\lim_{t\to\infty} \frac{1}{t^{p-1}}$ exists and if $0 < p < 1$, then $\lim_{t\to\infty} \frac{1}{t^{p-1}}$ does not exist which implies $\int_1^{\infty} f(x)dx$ converges if $p > 1$ and diverges if $0 < p < 1$.

Hence the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $0 < p \leq 1$.

Example 11: Test the convergence or divergence of the following series using the P- test.

a. $\sum_{n=1}^{\infty} \frac{1}{n}$

Solution: here $p = 1$, the p-series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and is called harmonic series.

b. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Solution: $p = 2 > 1$ and hence by the p - series test it converges.

c. $\sum_{n=1}^{\infty} \frac{1}{2n^{2/3}}$

Solution: $p = 2/3 < 1$, and, hence it diverges.

d. $\sum_{n=1}^{\infty} \frac{5}{n^{5/4}}$

Solution: $p = 5/4 > 1$, therefore it converges.

2.4.2. Comparison test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be non negative term series (or positive series)

i. If $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

ii. If $\sum_{n=1}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 12:

a. Show that $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges.

Solution: $\frac{1}{2^n+1} \leq \frac{1}{2^n}$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (why?).

Now let $a_n = \frac{1}{2^n+1}$ and $b_n = \frac{1}{2^n}$. Then we

- $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$ converges and
- $a_n \leq b_n$ for all n

Hence from the comparison test it follows that $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges.

b. Show that $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}-1}$

Solution: we know that $2\sqrt{n} - 1 \leq 2\sqrt{n}$. From which we obtain $\frac{1}{2\sqrt{n}} \leq \frac{1}{2\sqrt{n}-1}$ for all $n \geq 1$.

But $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a p series with $p = \frac{1}{2} < 1$, therefore, $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$ diverges.

Now let $a_n = \frac{1}{2\sqrt{n}-1}$ and $b_n = \frac{1}{2\sqrt{n}}$

- $0 \leq b_n \leq a_n$ for all n and
- $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$ diverges

Hence from the comparison test it follows that $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}-1}$ diverges.

c. Show that $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges.

Solution: $\frac{1}{2^n-1} \leq \frac{1}{2^n-2^{n-1}} = \frac{1}{2^{n-1}}$ for all $n \geq 1$ and since the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges, then by the comparison test the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges.

Remark 6:

In using comparison test, we must, of course, have some well known series $\sum_{n=1}^{\infty} b_n$ for the purpose of comparison. Most of the time we use one of the following series:

i. P-series i.e. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where $p > 0$

ii. Geometric series i.e. $\sum_{n=1}^{\infty} a_n r^{n-1}$

Example 13: Determine whether the following series converges or diverges.

a. $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+8}$

Solution: we know that $\frac{5}{2n^2+4n+8} \leq \frac{5}{2n^2}$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \frac{5}{2n^2}$ is a p series with $p = 2 > 1$, hence, converges. Then by comparison test $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+8}$ converges.

b. $\sum_{n=1}^{\infty} \frac{2n}{3n^3-1}$

Solution: for $n \geq 1$, $n^3 \geq 1$ and $3n^3 - 1 \geq 2n^3$; consequently we have $\frac{2n}{3n^3-1} \leq \frac{2n}{2n^3} = \frac{1}{n^2}$.

From the p-series test, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Thus by comparison test $\sum_{n=1}^{\infty} \frac{2n}{3n^3-1}$ converges.

Theorem 6: (limit comparison test):

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be non negative term series (or positive series). Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is a positive number.

a. If $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$.

b. If $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 14: Test the convergence or divergence of the following series using limit comparison test.

a. $\sum_{n=1}^{\infty} \frac{4n-3}{n^3-5n-7}$

Solution: consider a series $b_n = \frac{4n}{n^3} = \frac{4}{n^2}$. Moreover, we know that $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{4}{n^2}$ converges and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{4n-3}{n^3-5n-7}}{\frac{4}{n^2}} = \lim_{n\to\infty} \frac{4n^3-3n^2}{4n^3-20n-28} = \lim_{n\to\infty} \frac{4-3/n}{4-20/n^2-28/n^3} = 1$$

Thus by limit comparison test the series $\sum_{n=1}^{\infty} \frac{4n-3}{n^3-5n-7}$ converges.

b. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{8n^2-5n}}$

Solution: we disregard all but the highest power of n in the denominator and we obtain $\frac{1}{\sqrt[3]{8n^2}} = \frac{1}{2n^{2/3}}$.

But we know that $\sum_{n=1}^{\infty} \frac{1}{2n^{2/3}}$ is a p series with $p = \frac{2}{3} < 1$, hence diverges and

$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\frac{1}{\sqrt[3]{8n^2-5n}}}{\frac{1}{\sqrt[3]{8n^2}}} = \lim_{n\to\infty}\sqrt[3]{\frac{8n^2}{8n^2-5n}} = \lim_{n\to\infty}\sqrt[3]{\frac{8}{8-5/n}} = 1.$

Thus by the limit comparison test the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{8n^2-5n}}$ diverges.

2.4.3. The ratio test

Let $\sum_{n=1}^{\infty} a_n$ be non negative term series (or positive series). Assume that $a_n \neq 0$ for all n and that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = r(\text{possibly } \infty),$$

where r is a non negative number.

- **a.** If $0 \leq r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.
- **b.** If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- **c.** If $r = 1$, the test fails; we can't draw any conclusion about the convergence or divergence of the series.

Example 15: test the convergence or divergence of the following

a. $\sum_{n=1}^{\infty}\frac{2^n}{n!}$

Solution: $r = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\Big/\frac{2^n}{n!} = \lim_{n\to\infty}\frac{2.2^n.n!}{(n+1).n!.2^n} = \lim_{n\to\infty}\frac{2}{n+1} = 0$

Since $r = 0 < 1$, then the series $\sum_{n=1}^{\infty}\frac{2^n}{n!}$ converges by ratio test

b. $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$

Solution: we have $a_n = \frac{2^n}{n^2}$ and $a_{n+1} = \frac{2^{n+1}}{(n+1)^2}$

Now $r = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)^2}\Big/\frac{2^n}{n^2} = \lim_{n\to\infty}\frac{2^{n+1}.n^2}{(n+1)^2..2^n} = 2$.

Since $r = 2 > 1$, then the series $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$ diverges by ratio test.

c. $\sum_{n=1}^{\infty}\frac{n!}{n^n}$

Solution: we have $a_n = \frac{n!}{n^n}$ and $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$.

$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \Big/ \frac{n!}{n^n} = \frac{1}{e} < 1$. Thus the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

2.4.4. The root test

Let $\sum_{n=1}^{\infty} a_n$ be a non negative series and assume that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = (\lim_{n\to\infty} a_n)^{1/n} = r \text{ (possibly } \infty)$$

where r is a non negative number.

a. If $0 \le r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

b. If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

c. If $r = 1$, the test fails; we can't draw any conclusion about the convergence or divergence.

Example 16: Test the convergence or divergence of the following series.

a. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

Solution: $r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{2} = \lim_{n\to\infty} \frac{n^{1/n}}{2}$.

By using L'Hospital's rule we have $\lim_{n\to\infty} n^{1/n} = e^{\lim_{n\to\infty} \ln n^{1/n}} = 1$. Thus $r = \frac{1}{2} < 1$ implies the series $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.

b. $\sum_{n=1}^{\infty} (\frac{\ln n}{100})^n$

Solution: $r = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{(\frac{\ln n}{100})^n} = \lim_{n\to\infty} \frac{\ln n}{100} = \infty$ .Therefore the series diverges.

Exercises 2.4

1. *Test the convergence of following series using limit comparison test*

   a. $\sum_{n=1}^{\infty} \frac{1}{2n^3+1}$ *b.* $\sum_{n=1}^{\infty} \frac{1}{3n+1}$ *c.* $\sum_{n=1}^{\infty} \frac{n^3}{n^5+5n^2+7}$

2. *Test the convergence of following series using the root test*

   a. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$ *b.* $\sum_{n=1}^{\infty} \frac{2^n}{n^n}$ *c.* $\sum_{n=1}^{\infty} \left(\frac{n}{10}\right)^n$

3. *Test the convergence of following series using the ratio test*

   a. $\sum_{n=1}^{\infty}\frac{1}{n!}$ b. $\sum_{n=1}^{\infty}\frac{n^n}{n!}$ c. $\sum_{n=1}^{\infty}\frac{n}{10^n}$ d. $\sum_{n=1}^{\infty}\frac{n!}{2^n}$

4. *Test the convergence of following series using the integral test*

   a. $\sum_{n=1}^{\infty}\frac{1}{2n+1}$ b. $\sum_{n=1}^{\infty}\frac{1}{n^p}$, *p is a constant* c. $\sum_{n=1}^{\infty}\frac{1}{4n^2}$

   d. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n+1}}$

2.5. Alternating Series and alternating series test

Sometimes all the elements of the sequence may be positive or negative and for such we have stipulated the convergence tests. But when the element are alternatively positive and negative, the series is named as alternative series and for this we introduce test for its convergence, and is derived from the series called as alternative series test.

Overview

In this section, we are going to discuss about alternating series and its properties. Moreover, we will also learn to determine the convergence and divergence of alternating series by using the test.

Section Objective

On the completion of this subtopic, students should be able to:

- define alternating series;
- determine whether or not a particular series is alternating series or not;
- determine whether a particular series is convergent or divergent;

Definition 8: If the terms in a series are alternatively positive and negative, then we call the series an alternating series; or else,

A series of the form $a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1}.a_n + \ ...$ or $\sum_{n=1}^{\infty}(-1)^{n+1}.a_n$

Or

$-a_1 + a_2 - a_3 + a_4 - a_5 + \cdots + (-1)^{n}.a_n + \ ...$ or $\sum_{n=1}^{\infty}(-1)^{n}.a_n$ where $a_n > 0$ for all $n \in N$ is called an alternating series.

Example 17: Here some examples of alternating series.

- **a.** $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}.\frac{1}{n}$
- **b.** $\sum_{n=1}^{\infty}(-1)^{n+1}.3^n = 3 - 9 + 27 - 81 + \cdots$
- **c.** $\sum_{n=1}^{\infty}(-1)^{n} = -1 + 1 - 1 + 1 - 1 + \cdots$

Alternating series test

Theorem 7: Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$. Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n} a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges. Moreover, if S_n and S are the n^{th} partial sum and the sum of the infinite series $S = \sum_{n=1}^{\infty}(-1)^{n+1} a_n$ respectively, then $|S - S_n| \leq a_{n+1}$ for all natural number.

Remark 7: $R_n = |S - S_n|$ is called the n^{th} error in approximating S by S_n .

Example 18: Test the convergence of the following series using an alternating series, if possible.

- **a.** $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$

Solution: From the series, we have $a_n = \frac{1}{n}$ and since $\frac{1}{n+1} \leq \frac{1}{n}$ for all $n \in N$, then $a_{n+1} \leq a_n$ for all $n \in N$.

$\Longrightarrow$ $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence of positive numbers and $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$

Therefore $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is convergent by alternating series test.

- **b.** $\sum_{k=1}^{\infty}(-1)^{k+1}.\frac{k}{k^2+1}$

Solution: $a_n = \frac{n}{n^2+1}$ which implies $f(x) = \frac{x}{x^2+1}$ on $[1, \infty)$

$\Longrightarrow f'(x) = \frac{x'.(x^2+1)-x.(x^2+1)'}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \leq 0$ for all $x \in [1, \infty)$.

$\Longrightarrow$ f(x) is decreasing on $[1, \infty)$

$a_{n+1} \leq a_n$ for all $n \in N$.

$\Longrightarrow \{a_n\}_{n=1}^{\infty}$ is a decreasing sequence of positive numbers and

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0.$

Thus by alternating series test $\sum_{k=1}^{\infty}(-1)^{k+1}.\frac{k}{k^2+1}$ converges.

c. $\sum_{n=5}^{\infty} \frac{(-1)^n}{\ln n}$

Solution: $a_n = \frac{1}{\ln n}$ is a decreasing non negative sequence and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln n} = 0$. Thus by alternating series test $\sum_{n=5}^{\infty} \frac{(-1)^n}{\ln n}$ converges.

d. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{1}{2n+2}$

Solution: $a_n = \frac{1}{2n+2}$ is a decreasing sequence of positive numbers and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2n+2} = 0$, thus, by alternating series test $\sum_{n=1}^{\infty}(-1)^{n+1}.\frac{1}{2n+2}$ converges.

e. $\sum_{n=1}^{\infty}(-1)^{n} \frac{7n+6}{10n+1}$

Solution: $a_n = \frac{7n+6}{10n+1}$ is a decreasing sequence of positive numbers and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{7n+6}{10n+1} = \frac{7}{10} \neq 0.$

Therefore, by divergent test, $\sum_{n=1}^{\infty}(-1)^{n}.\frac{7n+6}{10n+1}$ diverges.

f. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n+1}{4n}$

Solution: $a_n = \frac{n+1}{4n}$ is a decreasing sequence of positive numbers and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{4n} = \frac{1}{4} \neq 0$

Thus by divergent test $\sum_{n=1}^{\infty}(-1)^{n+1}. \frac{n+1}{4n}$ diverges.

g. $\sum_{n=1}^{\infty}(-1)^{n} \frac{n+2}{n^2 + 3n+5}$

Solution: $a_n = \frac{n+2}{n^2+3n+5}$ is a decreasing sequence of positive numbers and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+2}{n^2+3n+5} = 0$

Thus by alternating series test $\sum_{n=1}^{\infty}(-1)^n.\frac{n+2}{n^2+3n+5}$ converges.

h. $\sum_{n=1}^{\infty}(-1)^n \frac{\ln n}{n}$

Solution: $a_n = \frac{\ln n}{n}$ is a decreasing sequence of positive numbers and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$. Thus by alternating series test $\sum_{n=1}^{\infty}(-1)^n.\frac{\ln n}{n}$ converges

i. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n^2}{2n+1}$

Solution: $a_n = \frac{n^2}{2n+1}$ is not a decreasing sequence of positive numbers and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{2n+1} = \infty$. Thus by divergent test $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n^2}{2n+1}$ diverges.

j. $\sum_{n=1}^{\infty}(-1)^{n+1}.\frac{1}{n^{1/10}}$

Solution: $a_n = \frac{1}{n^{1/10}}$ is a decreasing sequence of positive numbers and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{1/10}} = 0$. Thus by alternating series test $\sum_{n=1}^{\infty}(-1)^{n+1}.\frac{1}{n^{1/10}}$ converges.

k. $\sum_{n=1}^{\infty}(-1)^n \frac{(\ln n)^2}{n}$

Solution: $a_n = \frac{(\ln n)^2}{n}$ is a decreasing sequence of positive numbers and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{n\to\infty} \frac{\ln n}{n} = 0$

Thus by alternating series test $\sum_{n=1}^{\infty}(-1)^n.\frac{(\ln n)^2}{n}$ converges

Exercises 2.5

1. *Test the converges of the following alternating series*

   a. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ *b.* $\sum_{n=1}^{\infty}(-1)^{n+1}\, n$ *c.* $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2+1}$

   d. $\sum_{n=1}^{\infty}(-1)^{n+1}.sin\left(\frac{2}{n}\right)$

2. *Given the series* $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$

   a. *Prove the series converges*

b. *Find the maximum error made in approximating the sum of the first 8 terms ?*

c. *How many terms of the series are needed in order to obtain an error which does not exceed 0.01 ?*

2.6. Absolute and conditional convergence

Where the terms of the series are positive, we can easily determine the convergence or divergence of the series by using one of the fore mentioned convergence tests; however, if the terms are negative or a series with positive and negative terms, we need to devise a convergence tests with which we determine the convergence or divergence of the series. But most often this is too difficult. In such a case, we can specify the need of absolute or conditional convergence.

Overview

In this section, we are going to discuss about absolute and conditional convergence, and the test of convergence associated with.

Section Objective

On the completion of this subtopic, students should be able to:

- define absolute convergence;
- define conditional convergence;
- determine series which converge absolutely, conditionally or neither.

Definition 9: A convergent series $\sum_{n=1}^{\infty} a_n$ is said to be

i. absolute convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

ii. conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Remark 8:

1. When the series $\sum_{n=1}^{\infty} a_n$ converges, the series $\sum_{n=1}^{\infty} |a_n|$ may or may not converge.
2. All convergent non negative (positive) series converges absolutely.

Example 19: Determine which series converges absolutely, converges conditionally or diverges.

a. $\sum_{n=1}^{\infty}(-1)^n.\frac{1}{n^2}$

Solution: $a_n = \frac{1}{n^2}$ is a decreasing sequence of positive numbers and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^2} = 0$. Then by alternating series test the series $\sum_{n=1}^{\infty}(-1)^n.\frac{1}{n^2}$ converges.

Consider the absolute value of the series,

$$\sum_{n=1}^{\infty}\left|(-1)^n.\frac{1}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

is a p series with $p = 2 > 1$, hence, it converges. Therefore $\sum_{n=1}^{\infty}(-1)^n.\frac{1}{n^2}$ is absolutely convergent.

b. $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \frac{1}{2^4} - \frac{1}{2^5} + \cdots$

Solution: The series is defined by $\sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n}$. From which we have $a_n = \frac{1}{2^n}$ and, clearly it is a decreasing sequence of positive numbers. Moreover, we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^n} = 0.$

Therefore, by alternating series test, the series $\sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n}$ converges.

Consider, then, the absolute value of the series, that is,

$$\sum_{n=0}^{\infty}\left|(-1)^n.\ \frac{1}{2^n}\right| = \sum_{n=1}^{\infty}\frac{1}{2^n}$$

is a geometric series with $r = \frac{1}{2} < 1$; hence, it converges.

Therefore $\sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n}$ is absolutely convergent.

c. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

Solution: we have $a_n = \frac{1}{n}$ and is a decreasing sequence of positive numbers and $\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{n} = 0$. Thus by alternating series test $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges.

The absolute value of the series

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$$

is a harmonic series which diverges.

Therefore $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges conditionally.

d. $\sum_{n=1}^{\infty} (-1)^{n+1} . \frac{1}{3n+5}$

Solution: The series is an alternating series and $a_n = \frac{1}{3n+5}$ is decreasing sequence of positive numbers. Moreover, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{3n+5} = 0$. Thus by alternating series test $\sum_{n=1}^{\infty} (-1)^{n+1} . \frac{1}{3n+5}$ converges.

The absolute value of the series is $\sum_{n=1}^{\infty} \left|(-1)^{n+1} . \frac{1}{3n+5}\right| = \sum_{n=1}^{\infty} \frac{1}{3n+5}$.

Using an integral test $f(x) = \frac{1}{3x+5}$ is continuous and decreasing on $[1, \infty)$.

$$\sum_{n=1}^{\infty} \frac{1}{3n+5} = \lim_{b\to\infty} \int_1^b \frac{1}{3x+5} dx = \lim_{b\to\infty} \frac{1}{3} . \ln|3x+5| |_1^b$$

$$= \frac{1}{3} . \lim_{b\to\infty} \ln|3b+5| - \ln 8 = \infty$$

$\sum_{n=1}^{\infty} \left|(-1)^{n+1} . \frac{1}{3n+5}\right|$ diverges .

Therefore $\sum_{n=1}^{\infty} (-1)^{n+1} . \frac{1}{3n+5}$ is conditionally convergent.

e. $\sum_{n=1}^{\infty} (-1)^{n+1} . \frac{1}{n^{1/n}}$

Solution: We have $a_n = \frac{1}{n^{1/n}}$ is a decreasing sequence and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n^{1/n}}$.

$$= \lim_{n\to\infty} \frac{1}{e^{\frac{1}{n}\ln n}} = \frac{1}{\lim_{n\to\infty} e^{\frac{1}{n}\ln n}} = \frac{1}{e^{\lim_{n\to\infty} \frac{\ln n}{n}}} = 1 \neq 0$$

Thus by divergence test $\sum_{n=1}^{\infty} (-1)^{n+1} . \frac{1}{n^{1/n}}$ diverges.

Remark 9:

1. Every absolutely convergent series is convergent, that is, if $\sum_{n=1}^{\infty} |a_n|$ converges , then $\sum_{n=1}^{\infty} a_n$ converges.
2. $|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} |a_n|$
3. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent, then $\sum_{n=1}^{\infty} (a_n \pm b_n)$ and $\sum_{n=1}^{\infty} ca_n$ for $c \in R$ are absolutely convergent.

Theorem 8: Every absolutely convergent series is convergent. (If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.)

Proof: (omitted)

Exercises 2.6

1. *Determine whether the following series is absolutely convergent, conditionally convergent, and divergent?*

   a. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

   b. b. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

   c. c. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$

2.7. Generalized Convergence tests

Where the terms of the series are positive, we can easily determine the convergence or divergence of the series by using one of the fore mentioned convergence tests; however, if the terms are negative or a series with positive and negative terms, we need to devise a convergence tests with which we determine the convergence or divergence of the series. But most often this is too difficult. In such a case, we can specify the need of absolute or conditional convergence.

Overview

In this subtopic, we will see the various convergence tests for a series with positive and negative terms. We verify this by using various examples.

Section Objective

On the completion of this subtopic, students should be able to:

- explain the need of generalized tests;
- use generalized convergence tests to determine whether or not a particular series converge absolutely or conditionally.

Theorem 9 (Generalized convergence tests): Let $\sum_{n=1}^{\infty} a_n$ be a series

1. Generalized comparison tests

If $|a_n| \leq |b_n|$ for $n \geq 1$ and if $\sum_{n=1}^{\infty} |b_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges (absolutely).

2. Generalized limit comparison tests

If $\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = L$, where L is a positive number and if $\sum_{n=1}^{\infty} |b_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges (absolutely).

3. Generalized ratio test

Suppose that $a_n \neq 0$ for $n \geq 1$ and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r$ (possibly ∞)

- If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges .
- If $r = 1$, we cannot draw any conclusions from this test alone about the convergence of the series.

4. Generalized root test

Suppose that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r$ (possibly ∞)

- If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $r = 1$, we cannot draw any conclusions from this test alone about the convergence of the series.

Example 20: Test the convergence of the following series using the generalized convergence tests.

a. $\sum_{n=1}^{\infty}(-1)^n.\frac{n}{6^n}$

Solution: we have $a_n = \frac{n}{6^n}$ and using the generalized root test we obtain,

$$R = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{6^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}}{6} = \frac{\lim_{n\to\infty} e^{\frac{1}{n}\ln n}}{6} = \frac{1}{6} < 1 .$$

Therefore,

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{6^n}$$

converges absolutely.

Or;

Using generalized ratio test we have,

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{6^{n+1}}\Big/\frac{n}{6^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{6.6^n}.\frac{6^n}{n}\right|$$

$$= \frac{1}{6}.\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{1}{6} < 1$$

Therefore,

$$\sum_{n=1}^{\infty}(-1)^n.\frac{n}{6^n} \text{ converges absolutely.}$$

b. $\sum_{n=1}^{\infty}\frac{(-8)^n.n}{6^n}$

Solution: we have $a_n = \frac{(-8)^n.n}{6^n}$ and using the generalized root test we obtain

$$r = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left|\frac{(-8)^n.n}{6^n}\right|}$$

$$= \lim_{n\to\infty}\sqrt[n]{\left(\frac{8}{6}\right)^n.n} = \frac{8}{6}.\lim_{n\to\infty}\sqrt[n]{n} = \frac{8}{6} > 1$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{(-8)^n.n}{6^n} \text{ diverges.}$$

c. Show that $\sum_{n=1}^{\infty}\frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \ ...$

- **a.** Converges absolutely for $|x| < 1$
- **b.** Converges conditionally for $|x| = -1$
- **c.** Diverges for $x = 1$ and $|x| > 1$

Solution: If $x = 0$, the series $\sum_{n=1}^{\infty}\frac{x^n}{n}$ converges.

If $x \neq 0$, using the generalized ratio test we have,

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+1}}{n+1}.\frac{n}{x^n}\right| = \lim_{n\to\infty}\left|x.\frac{n}{n+1}\right| = |x|$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{x^n}{n} = \begin{cases}\text{converges for } |x| < 1\\ \text{diverges for } |x| > 1\end{cases}$$

For $x = 1$, $\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$ is a harmonic series which is divergent.

For $x = -1$, $\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ is an alternating series which converges conditionally.

d. For what value of x does the series $\sum_{n=1}^{\infty}\frac{(-1)^n . x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

- **a.** Absolutely convergent
- **b.** Conditionally convergent

c. Divergent

Solution:

Case I: if $x = 0$, the series converges .

Case II: if $x \neq 0$

Using the generalized ratio test we have

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\,.\,x^{2(n+1)+1}}{2(n+1)+1}.\ \frac{2n+1}{(-1)^n\,.\,x^{2n+1}}\right| =$$

$$= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\,.\,x^{2n+3}\,.2n+1}{(2n+3)\,.\,(-1)^n\,.\,x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|x^2\,.\,\frac{(2n+1)}{(2n+3)}\right| = |x^2| = x^2$$

Thus by the generalized ratio test, the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n\,.\,x^{2n+1}}{2n+1} = \begin{cases}\text{converges for } |x^2| < 1 \Rightarrow |x| < 1\\ \text{diverges for } |x^2| > 1 \Rightarrow |x| > 1\end{cases}$$

For $x = -1$, $\sum_{n=1}^{\infty}\frac{(-1)^n\,.\,x^{2n+1}}{2n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n+1}$ is an alternating series, hence, the series converges by alternating series test.

For $x = 1$, $\sum_{n=1}^{\infty}\frac{(-1)^n\,.\,x^{2n+1}}{2n+1} = \sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1}$ is an alternating series which converges by an alternating series test.

Using integral test

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n+1} = \lim_{b\to\infty}\int_0^b\frac{1}{2x+1}\,dx = \lim_{n\to\infty}[\tfrac{1}{2}\,.\ln|2x+1|]|_0^b$$

$$= \lim_{b\to\infty}\frac{1}{2}\ln|2b+1| - 0 = \infty$$

Thus, the given series converges conditionally.

e. For what value of x does the series $\sum_{n=1}^{\infty}\frac{x^{n-1}}{n\,.\,3^n}$

a. Converges absolutely

b. Converges conditionally

c. Diverges

Solution:

Case I: if $x = 0$, the series converges

Case II: if $x \neq 0$

Using the generalized ratio test we have

$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^n}{(n+1)\,.\,3^{n+1}}.\ \frac{n\,.\,3^n}{x^{n-1}}\right| = \ = \lim_{n\to\infty}\left|\frac{x}{3}\,.\ \frac{n}{n+1}\right| = \left|\frac{x}{3}\right|$

Thus by the generalized ratio test, the series

i. Converges absolutely for $\left|\frac{x}{3}\right| < 1$ which implies $|x| < 3$

ii. Diverges for $\left|\frac{x}{3}\right| > 1$ which implies $|x| > 3$

For $x = -3$, series becomes $\sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{n\,.\,3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ is an alternating series and , hence, converges by alternating series test.

But $\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n}$ is harmonic series and diverges.

For $x = 3$, the series becomes $\sum_{n=1}^{\infty}\frac{3^{n-1}}{n\,.\,3^n} = \sum_{n=1}^{\infty}\frac{1}{3n} = \frac{1}{3}\,.\sum_{n=1}^{\infty}\frac{1}{n}$ which is a harmonic divergent series.

Therefore the series converges absolutely if $|x| < 3$; converges conditionally $x = -3$ and diverges $|x| > 3$ and $x = 3$.

Exercises 2.7

1. *For what value of x does the following series*

i. *Converges absolutely*

ii. *Conditionally converges*

iii. *Diverges*

a. $\sum_{n=1}^{\infty}\frac{(x-1)^n}{2n+1}$ *b.* $\sum_{n=1}^{\infty}\frac{n.(x-1)^n}{(3n-1).2^n}$ *c.* $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}.\ x^{n-1}}{(2n-1)!}$

d. $\sum_{n=1}^{\infty}\frac{1}{2n-1}\,.\left(\frac{x+2}{x-1}\right)^n$

Unit Summary:

1) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, then the expression $a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ which is denoted by $\sum_{i=1}^{\infty} a_i$, that is, $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$ is called an infinite series.
2) Consider the series $\sum_{i=1}^{\infty} a_i = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$, then
   - a_n is called the n^{th} - term of the series and $a_n = S_n - S_{n-1}$.
   - Let $S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \ldots + a_n$, then S_n is called the n^{th} - partial sum of the series.
   - $\{S_n\}_{n=1}^{\infty}$ where S_n is the n^{th} – partial sum is called the sequence of partial sums.
3) **Convergent series**. An infinite series $\sum_{n=1}^{\infty} a_n$ with sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ is said to be convergent if and only if the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ converges, i.e., if $\lim_{n\to\infty} s_n$ exists, then we say that the series $\sum_{n=1}^{\infty} a_n$ is a convergent series and we write it as $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$.
4) **Divergent series**. A series $\sum_{n=1}^{\infty} a_n$ is said to be divergent if it is not convergent, i.e., the series $\sum_{n=1}^{\infty} a_n$ is divergent if and only if the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ is divergent.
5) **Divergence test (The n^{th} – term test)**. If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
6) **Non negative series:** If $a_n \geq 0$ (non negative) for every positive integer n, then the series $\sum_{n=1}^{\infty} a_n$ is called a non negative series.
7) **Positive Series:** If $a_n > 0$ (positive) for every positive integer n, then the series $\sum_{n=1}^{\infty} a_n$ is called a positive term series or positive series.
8) There are two types of convergence tests: one that compares a non negative series with an improper integral and one that compares a given non- negative (positive) series with another series.
   - **The integral test :** Let $\{a_n\}_{n=1}^{\infty}$ be a non- negative sequence and f be a continuous decreasing function defined on $[1, \infty)$ such that $f(n) = a_n$ for

all $n \geq 1$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the improper integral $\int_1^{\infty} f(x)dx$ converges.

- **The p-series:** The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ where p a real number is called a p series.

Theorem.(**p-series test**) The p- series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

i. Converges if $p > 1$

ii. Diverges if $p \leq 1$.

- **Comparison test:** Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be non negative term series (or positive series)

i. If $\sum_{n=1}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

ii. If $\sum_{n=1}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$ for all $n \geq 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

- **The root test :** Let $\sum_{n=1}^{\infty} a_n$ be a non negative series and assume that $\lim_{n\to\infty} \sqrt[n]{a_n} = (\lim_{n\to\infty} a_n)^{1/n} = r$ (possibly ∞) , where r is a non negative number

a. If $0 \leq r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

b. If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

c. If $r = 1$, the test fails; we can't draw any conclusion about the convergence or divergence.

- **The ratio test :** Let $\sum_{n=1}^{\infty} a_n$ be non negative term series (or positive series). Assume that $a_n \neq 0$ for all n and that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$(possibly ∞) , where r is a non negative number.

a. If $0 \leq r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

b. If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

c. If $r = 1$, the test fails; we can't draw any conclusion about the convergence or divergence of the series.

11) If the terms in a series are alternatively positive and negative, then we call the series an alternating series; or else,

- A series of the form $a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1}.a_n + \ldots$ or $\sum_{n=1}^{\infty}(-1)^{n+1}.a_n$. Or,

- $-a_1 + a_2 - a_3 + a_4 - a_5 + \cdots + (-1)^n . a_n + \ldots$ or $\sum_{n=1}^{\infty}(-1)^n . a_n$ where $a_n > 0$ for all n ∈ N is called an alternating series

Alternating series test: Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$. Then the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converges. Moreover, if S_n and S are the n^{th} partial sum and the sum of the infinite series $S = \sum_{n=1}^{\infty}(-1)^{n+1} a_n$ respectively, then $|S - S_n| \le a_{n+1}$ for all natural number.

12) A convergent series $\sum_{n=1}^{\infty} a_n$ is said to be

i. absolute convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.

ii. conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges

13) Generalized convergence tests: Let $\sum_{n=1}^{\infty} a_n$ be a series

1. Generalized comparison tests

If $|a_n| \le |b_n|$ for $n \ge 1$ and if $\sum_{n=1}^{\infty} |b_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges (absolutely).

2. Generalized limit comparison tests

If $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = L$, where L is a positive number and if $\sum_{n=1}^{\infty} |b_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges (absolutely).

3. Generalized ratio test

Suppose that $a_n \neq 0$ for $n \ge 1$ and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = r$ (possibly ∞)

- If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges .
- If $r = 1$, we cannot draw any conclusions from this test alone about the convergence of the series.

4. Generalized root test

Suppose that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r$ (possibly ∞)

- If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
- If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $r = 1$, we cannot draw any conclusions from this test alone about the convergence of the series.

Miscellaneous Exercises

1. *If the* n^{th} *partial sum of the series is given by* $S_n = \frac{n-1}{n+1}$, *then find the*
   a. *Third term of the sequence i.e.* a_3
   b. n^{th} *Term of the sequence i.e.* a_n
   c. $\sum_{n=1}^{\infty} a_n$
2. *Show that*
   a. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges then* $\lim_{n\to\infty} a_n = 0$
   b. *If a series converges absolutely then it converges*
   c. *If a sequence converges then it bounded*
   d. *If a sequence is bounded and increasing then it converges*
3. *Find the following sums*
   a. $\sum_{n=3}^{\infty} \frac{1}{n(n+1)}$ **b.** $\sum_{n=1}^{\infty} \left(\frac{2}{4n^2-1} - \frac{1}{2^n}\right)$ **c.** $\sum_{n=0}^{\infty} (-1)^n \frac{3^{n+3}}{5^{n-1}}$
4. *Determine whether the series converges or diverges.*
   a. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ **b.** $\sum_{n=1}^{\infty} \left(\frac{1+2n}{n}\right)^n$ **c.** $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$
5. *Find the general term of the sequence* $\left\{1, \frac{-2}{3}, \frac{4}{9}, \frac{-8}{27}, \ldots\ldots\right\}$; $a_n =$ ______________
6. *If the* n^{th} *partial sum of the series is given by* $S_n = 2n^2 + 1$, *then find the*

   a. *Third term of the sequence i.e.* a_3 ?
   b. n^{th} *term of the sequence i.e.* a_n ?
7. *Which of the following is false?*
   a. *If the series* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.
   b. *If* $\lim_{n\to\infty} a_n$ *does not exist or* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *is divergent.*
   c. *If* $\lim_{n\to\infty} a_n = 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *converges.*

Justify by giving a counter example: ______________________________

8. *Test the convergence of the series* $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.
9. *Show that* $\sum_{n=3}^{\infty} \frac{1}{n(n+1)}$ *converges and find its sum.*
10. *Define absolute and conditional convergence.*

11. *Find value c such that the series* $\sum_{n=1}^{\infty}(c+2)^n$ *= 3.*

12. *Use integral test to determine convergence or divergence of the series* $\sum_{n=1}^{\infty}\frac{e^n}{e^{2n}+1}$.

13. *Using partial sum determine whether the series* $\sum_{n=1}^{\infty}\frac{n}{2}$ *converges or diverges.*

14. *Consider the series* $\sum_{n=1}^{\infty}\frac{n}{(n+1)!}$

   a. *Find partial sum* S_n *by first finding partial sums* $S_1,\ S_2, S_3$ *and* S_4*.*

   b. *Find the sum of the series.*

15. *Determine the value of* x *such that the series* $\sum_{n=1}^{\infty}\frac{x^{2n}}{n}$ *converges and for what value of* x *it diverges.*

16. *Given a series* $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+1}$

   a. *Determine whether the series absolutely converges or not.*

   b. *Show that it is convergent series.*

17. *Every series which is not absolutely convergent is divergent.* ***(True/False)*** *If false, give a counter example.*

18. *Determine whether the series* $5-\frac{10}{3}+\frac{20}{9}-\frac{40}{27}+\ \dots$ *is convergent or divergent. If it is convergent, find its sum.*

19. *Find the value of* x *for which the series* $\sum_{n=0}^{\infty}\left(\frac{3x}{2+x^2}\right)^{2n}$ *converges.*

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Chapter Three

Power Series

Introduction

The purpose of this chapter is to give a systematic exposition of some of the most important thing about a power series.

The representation of functions of power series is one of the most useful of mathematical techniques in a wide variety of situations. Sometimes we start from a function that is defined for us in some manner not employing series, and seek to expand the function in a power series. In either of these situations, we need to know something of what properties a function has if it is defined by a power series.

This chapter begins with a statement of what is meant by power series, then the question of when these sums can be assigned values is addressed. Much information can be obtained by exploring infinite sums of constant terms; however, the eventual objective is to introduce series that depend on variables. This presents the possibility of representing functions by series. Afterwards, the question of how continuity, differentiability, and integrabiltiy play a role can be examined.

Unit Objectives:

On the completion of this unit, students should be able to:

- understand the definition of a power series,
- define the convergence and divergence of power series,
- find radius of convergence of a power series,
- find the limit of convergent power series,
- represent a wide class of functions by a Taylor's series,
- apply Taylor's polynomial,
- approximate a function by a Taylor polynomial.
- find the derivative of a power series
- find the integral of a power series

3.1. Definition of Power Series

Overview

In the previous sections we focused exclusively on series whose terms are numbers. In this section we will consider series whose terms are functions with the objective of developing the mathematical tools needed to investigate the convergence of Taylor and Maclaurin series.

Section objective:

After you complete the study of this subtopic, you should be able to:

- define the power series;
- give an example of power series; and realize the need for power series.
- find a power series

The general form of a power series is

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n + \cdots \qquad (3.1)$$

this is called a power series in $(x - x_0)$. Hence x_0 is fixed and x is variable. In the special case where $x_0 = 0$ the series takes the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \qquad (3.2)$$

It turns out that in studying power series it is sufficient to consider (3.2), since the general case (3.1) can be reduced to (3.2) by a translation of the origin along the axis. For this reason, all the general theory of power series here after will be developed for series of power series of x, of the form (3.2).

Definition 1: A series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \quad \text{or} \quad \sum_{n=0}^{\infty} a_nx^n$$

is called a power series in x or a power series.

Or, a more generalized form of a power series in $(x - a)$, that is, an infinite series of the form

$$a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n + \cdots \quad \text{or} \quad \sum_{n=0}^{\infty} a_n(x - a)^n$$

is called a power series in $(x - a)$.

If $a = 0$, this general power series becomes a power series in x

Remark 1:

1. The initial index of a power series can be any non negative numbers, that is, for the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$.

   If $a_0 = 0$, the power series is given by $\sum_{n=1}^{\infty} a_n x^n$.

   If $a_1 = 0$, the power series is given by $\sum_{n=2}^{\infty} a_n x^n$.

2. In the power series $\sum_{n=0}^{\infty} a_n x^n$, a_n represents the constant coefficients and x is the variable.

Example 1: The following are some examples of power series.

a. $\sum_{n=0}^{\infty} 2x^n = 2 + 2x + 2x^2 + \cdots = 2(1 + x + x^2 + \cdots)$ is a power series with coefficient 2 and centre at 0.

b. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ is power series with coefficient $a_i's = 1$ and centre at 0. Here the power series is geometric series.

c. $\sum_{n=0}^{\infty} n(x-2)^n = (x-2) + 2(x-2)^2 + 3(x-2)^3 + \cdots$ is a power series with centre at 2 and coefficient $a_n = n$.

d. $\sum_{n=6}^{\infty} \frac{1}{n-5}(x-7)^n = (x-7)^6 + \frac{1}{2}(x-7)^7 + \frac{1}{3}(x-7)^8 +$ ... is a power series with coefficient $a_n = \frac{1}{n-5}$ and centre 7.

Example 2:

1. Show that $\frac{1}{1-x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < 1$

Solution. First we consider the power series:

$$1 + x + x^2 + x^3 + x^4 + \cdots$$

This is a geometric series with ratio x. Therefore, it converges for $|x| < 1$. The sum of the series is $\frac{1}{1-x}$. Substituting $-x$ for x, we have

$$1 - x + x^2 - x^3 + x^4 - x^5 + \cdots = \frac{1}{1-(-x)} = \frac{1}{1+x}, \qquad |x| < 1$$

Thus,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

2. Find a power series for the rational fraction $\frac{1}{2-x}$.

Solution. We can write this function as $\frac{1}{2-x} = \frac{1/2}{1-x/2}$. As can be seen, this is the sum of the infinite geometric series with the first term $\frac{1}{2}$ and ratio $\frac{x}{2}$:

$\frac{1}{2}+\frac{1}{2}\frac{x}{2}+\frac{1}{2}(\frac{x}{2})^2+\frac{1}{2}(\frac{x}{2})^3+\cdots=\frac{1}{2}+\frac{x}{2^2}+\frac{x^2}{2^3}+\frac{x^3}{2^4}+\cdots=\sum_{n=0}^{\infty}\frac{x^n}{2^{n+1}}$

The given power series converges for $|x| < 2$.

1. Find a power series for $\frac{6x}{5x^2-4x-1}$

Solution. First we find the partial fraction decomposition of this function. The quadratic function in the denominator can be written as $5x^2-4x-1=(5x+1)(x-1)$, so we can set:

$\frac{6x}{5x^2-4x-1}=\frac{A}{5x+1}+\frac{B}{x-1}$.

Multiply both sides of the expression by $5x^2-4x-1=(5x+1)(x-1)$ to obtain

$6x = A(x-1)+B(5x+1)$,

$\Longrightarrow 6x = Ax - A + 5Bx + B,$

$\Longrightarrow 6x = (A+5B)x + (-A+B),$

$\Longrightarrow \begin{cases} A+5B=6 \\ -A+B=0 \end{cases}$

The solution of this system of equations is $A = 1, B = 1$. Hence, the partial decomposition of the given function is

$\frac{6x}{5x^2-4x-1}=\frac{1}{5x+1}+\frac{1}{x-1}=\frac{1}{5x+1}-\frac{1}{1-x}$

Both fractions are the sums of the infinite geometric series:

$\frac{1}{5x+1}=\frac{1}{1-(-5x)}=1-5x+(-5x)^2+(-5x)^3+\cdots=\sum_{n=0}^{\infty}(-5x)^n$,

$\frac{1}{x-1}=1+x+x^2+x^3+\cdots=\sum_{n=0}^{\infty}x^n$

Hence, the expansion of the initial function is

$\frac{6x}{5x^2-4x-1}=\sum_{n=0}^{\infty}(-5x)^n-\sum_{n=0}^{\infty}x^n=\sum_{n=0}^{\infty}[(-5x)^n-x^n]=\sum_{n=0}^{\infty}[(-5)^n-1]x^n$.

Exercise 3.1

1. *Define a power series with center a, where a is real number such that*
   - ***(i).*** $a=0$
   - ***(ii).*** $a=5$, *coefficient* $a_n=\frac{1}{7}$
   - ***(iii).*** $a=5$, *coefficient* $a_n=\frac{1}{7}$
2. *Determine the center and coefficient of the following power series.*

   a. $\sum_{n=9}^{\infty}\frac{1}{n-2}(x-9)^n$ *b.* $\sum_{n=2}^{\infty}n^2x^n$

3.2. Convergence and divergence, Radius of convergence of a Power Series

Overview

Conceivable a particular power series may be convergent for all values of x; or else, it may also not be convergent for any values of x except the one value x.

In this section we study the convergence and divergence of power series, and also we will be able to determine whether or not a given power series is convergent or divergent.

Section objective:

After you complete the study of this subtopic, you should be able to:

- define convergent and divergent power series;
- determine convergent and divergent power series;
- find the radius and interval of convergence of a power series;

Definition 2: A power series is said to converge at x_0 if the series of real numbers $\sum_{n=0}^{\infty} a_n x^n$ converges at x_0 ; or,
A power series is said to be convergent in a set D of real numbers if it is convergent for every real number x in D.

Remark 2:

2. Every power series automatically converges for $x = 0$.
3. If $x \neq 0$, the series may or may not be convergent; we have to determine this using one of the convergence tests but the generalized ratio test, though not for all, is recommended.

Example 3: Determine whether or not the following power series are convergent or divergent.

a. $\sum_{n=0}^{\infty} n!\,x^n = 1 + x + 2!\,x^2 + 3!\,x^3 + \cdots = 1 + x + 2x^2 + 6x^3 + \ ...$

Solution:

Case I: If $x = 0$, the series becomes $\sum_{n=0}^{\infty} n!\,0 = \sum_{n=1}^{\infty} 0 \ = 0$ .The power series $\sum_{n=0}^{\infty} n!\,x^n$ converges at $x = 0$.

Case II: If $x \neq 0$, using the generalized ratio test,

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!\,.\,x^{n+1}}{n!\,.\,x^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1).\,n!\,.\,x.\,x^n}{n!\,.\,x^n}\right|$$

$$= \lim_{n\to\infty}|(n+1).\,x| = \infty$$

Then the power series diverges for any $x \neq 0$.

b. $\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} +$ ...

Solution: Case I: if $x = 0$, the power series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges at $x = 0$.

Case II: if $x \neq 0$, we have $a_n = \frac{x^n}{n!}$ and $a_{n+1} = \frac{x^{n+1}}{(n+1)!}$

Using the generalized ratio test

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\Big/\frac{x^n}{n!}\right| = \lim_{n\to\infty}\left|\frac{x\,.\,x^n.\,n!}{(n+1).\,n!\,.\,x^n}\right| = \lim_{n\to\infty}\left|\frac{x}{(n+1)}\right| = 0 < 1$$

The power series converges for any x.

c. $\sum_{n=0}^{\infty}x^n = 1 + x + x^2 + x^3 +$ ...

Solution: Case I: if $x = 0$, the power series $\sum_{n=0}^{\infty}x^n$ converges .

Case II: when $x \neq 0$, we have $a_n = x^n$ and $a_{n+1} = x^{n+1}$

Using the generalized ratio test

$$r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| = \lim_{n\to\infty}\left|\frac{x\,.\,x^n}{x^n}\right| = \lim_{n\to\infty}|x|$$

Now by using the generalized ratio test, the power series $\sum_{n=0}^{\infty}x^n$

- Converges for $r = |x| < 1$
- Diverges for $r = |x| > 1$ and
- For $|x| = 1$, the series becomes $\sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}1^n = \sum_{n=0}^{\infty}1$ diverges by divergent test.
- For $|x| \neq -1$, the series becomes $\sum_{n=0}^{\infty}x^n = \sum_{n=0}^{\infty}(-1)^n$ diverges

Therefore the power (geometric) series $\sum_{n=0}^{\infty}x^n$ converges for any $|x| < 1$ and diverges for $|x| \geq 1$.

Radius and interval of convergence

Theorem 1: Let $\sum_{n=0}^{\infty} a_n . x^n$ be a power series, then exactly one of the following conditions holds.

i. The power series $\sum_{n=0}^{\infty} a_n . x^n$ converges only at $x = 0$. Example: $\sum_{n=0}^{\infty} n!. x^n$

ii. The power series $\sum_{n=0}^{\infty} a_n . x^n$ converges for all x . Example: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

iii. There exists a positive real number R such that the power series $\sum_{n=0}^{\infty} a_n . x^n$

- Converges for all x with $|x| < R$, that is, $-R < x < R$.
- Diverges for all x with $|x| > R$

Definition 3:(radius of convergence) the number R in part (iii) is called the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n . x^n$ or the positive real number R such that the power series $\sum_{n=0}^{\infty} a_n . x^n$ converges for all x with $|x| < R$ or diverges for all with $|x| > R$.

Remark 3:

Every power series has a radius of convergence R which is either non-negative or ∞

$\Longrightarrow$ R always exists and is always non negative number

Definition 4: (Interval of convergence): The totality (collection) of values of x for which the power series $\sum_{n=0}^{\infty} a_n . x^n$ converges is called the interval of convergence of $\sum_{n=0}^{\infty} a_n . x^n$.

Remark 4: the interval of convergence takes one and only one of the following forms: $[0,0] = 0$, $[-R, R]$, $[-R, R)$, $(-\infty , \infty)$, $(-R , R)$, $(-R, R]$.

How to find the radius of convergence and interval of convergence

Step 1: Find r , i.e., using the generalized ratio test or root test

Consider a power series $\sum_{n=0}^{\infty} a_n . x^n$ and $r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, then

Step 2: determine the radius and interval of convergence

i. If $r = \infty$, then
   - Radius of convergence $R = 0$.
   - Interval of convergence $[0,0] = 0$,i.e., converges only at $x = 0$.

ii. If $r = 0$, then
   - Radius of convergence $R = \infty$
   - Interval of convergence $[-\infty, \infty]$,i.e., converges for all x

iii. If $0 < r < \infty$, then

- Radius of convergence $R = \frac{1}{r}$
- Interval of convergence $|x| < R$

Step 3: Test the end points $x = R$ or $x = -R$ for convergence, then determine if these boundary points are included in the interval of convergence or otherwise.

Example 4: Determine the radius of convergence (R) and interval of convergence for the following power series

a. $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

Solution: $\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} (\frac{x}{2})^n$

Step1: find r

Using the generalized root test $r = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{(\frac{x}{2})^n} = \frac{|x|}{2}$

Step2: by using the root test, the power series converges if $r = \frac{|x|}{2} < 1 \Rightarrow |x| < 2$

Thus the radius of convergence is $R = 2$

Step3: Check at the boundary : $x = 2$ or $x = -2$

For $x = 2$, the power series becomes $\sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1$ is divergent.

For $x = -2$, the power series $\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n$ is divergent.

The power series $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

- Radius of convergence is $R = \frac{1}{r} = 2$
- Interval of convergence $|x| < R \Rightarrow |x| < 2$ or $(-2, 2)$.

b. $\sum_{n=0}^{\infty} \frac{x^n}{n^{1/2}}$

Solution: find r , using the generalized ratio test

$$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)^{\frac{1}{2}}}}{\frac{x^n}{n^{\frac{1}{2}}}}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}.n^{\frac{1}{2}}}{(n+1)^{\frac{1}{2}}.x^n}\right|$$

$$= \lim_{n\to\infty} \left|\frac{x.\ x^n}{x^n} . (\frac{n}{n+1})^{\frac{1}{2}}\right| = \lim_{n\to\infty} \left|x . (\frac{n}{n+1})^{\frac{1}{2}}\right| = |x|$$

Step2: Determine the convergence by using the ratio test, the power series converges for $r = |x| < 1$ and diverges for $r = |x| > 1$.

Step 3: Check at the boundary

For $x = 1$, the power series becomes $\sum_{n=0}^{\infty} \frac{1}{n^{1/2}}$ diverges.

For $x = -1$, the power series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^{1/2}}$ is an alternating series that converges conditionally.

The power series $\sum_{n=0}^{\infty} \frac{x^n}{n^{1/2}}$

- Radius of convergence is $R = 1$
- Interval of convergence $[-1 , 1)$

c. $\sum_{n=0}^{\infty} n. x^n$

Solution: using the generalized ratio test

$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1).x^{n+1}}{n . x^n}\right| = \lim_{n\to\infty} \left|x. \frac{(n+1)}{n}\right| = |x|$

Determine the convergence by using the ratio test, the power series converges for $r = |x| < 1$ and diverges for $r = |x| > 1$.

Then check at the boundary points:

For $x = 1$, the power series becomes $\sum_{n=0}^{\infty} n = 1 + 2 + 3 + \cdots$ diverges.

For $x = -1$, the power series becomes $\sum_{n=0}^{\infty} (-1)^n . n$ is an alternating series that diverges.

Thus

- Radius of convergence is $R = 1$
- Interval of convergence $(-1,1)$

d. $\sum_{n=0}^{\infty} n!. x^n$

Solution: using the generalized ratio test

$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!.x^{n+1}}{n! . x^n}\right| = \lim_{n\to\infty} |n. x| = \infty$

Which implies converges only at $x = 0$

- Thus Radius of convergence is $R = 0$
- Interval of convergence $[0,0] = 0$

e. $\sum_{n=0}^{\infty} x^n$

Solution: using the generalized root test

$r = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{|x^n|} = |x|$

The power series converges if $r = |x| < 1$ and diverges for $r = |x| > 1$

Thus the radius of convergence is $R = 1$

Check at the boundary i.e. $x = 1$ or $x = -1$

For $x = 1$, the power series becomes $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 1$ is divergent.

For $x = -1$, the power series $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} (-1)^n$ is divergent.

Hence

- Radius of convergence is $R = 1$
- Interval of convergence $(-1, 1)$.

f. $\sum_{n=1}^{\infty} \frac{(-x)^n}{n}$

Solution: using the generalized ratio test

$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-x)^{n+1} .\ n}{(n+1)\,.\,(-x)^n}\right| = \lim_{n\to\infty} \left|\frac{n}{n+1} . -x\right| = |x|$

The power series $\sum_{n=1}^{\infty} \frac{(-x)^n}{n}$ converges for $r = |x| < 1$ and diverges for $r = |x| > 1$

For $x = 1$, the power series becomes $\sum_{n=1}^{\infty} \frac{(-x)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is an alternating series and converges conditionally.

For $x = -1$, the power series $\sum_{n=1}^{\infty} \frac{(-x)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is a harmonic series and diverges.

Therefore,

- Radius of convergence is $R = 1$
- Interval of convergence $(-1, 1]$

g. $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

Solution: using the generalized ratio test

$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1} .\ n!}{(n+1)!\,.\,x^n}\right| = 0 < 1$

$r = 0$, implies the power series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges for every x .

Therefore ,

- Radius of convergence is $R = \infty$
- Interval of convergence $(-\infty, \infty)$

h. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} . x^{2n+1}$

Solution: using the generalized ratio test

$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1} . x^{2n+3} .(2n+1)}{(-1)^n . x^{2n+1} .2n+3}\right| = |x^2| = x^2$

The power series converges for $r = x^2 < 1 \implies |x| < 1$ and diverges for $r = x^2 > 1 \implies |x| > 1$

For $x = 1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} . x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ is an alternating series and converges conditionally.

For $x = -1$, $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} . x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}$ is an alternating series and converges conditionally.

Thus,

- Radius of convergence is $R = 1$
- Interval of convergence $[-1,1]$

Exercise 3.2

1. *For each of the following series, find the radius of convergence(R) and the interval of convergence*

   i. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} . x^n}{4^n}$ *ii.* $\sum_{n=1}^{\infty} \frac{x^n}{n.2^n}$ *iii.* $\sum_{n=0}^{\infty} n! . \left(\frac{x}{2}\right)^2$

   iv. $\sum_{n=1}^{\infty} \frac{3.x^n}{3^n+5}$ *v.* $\sum_{n=0}^{\infty} \frac{(-3)^n . x^n}{\sqrt{n+1}}$ *vi.* $\sum_{n=0}^{\infty} \frac{n . (x+2)^n}{3^{n+1}}$

2. *Suppose that the radius of convergence of the power series* $\sum_{n=0}^{\infty} c_n . x^n$ *is R, what is the radius of convergence of the power series* $\sum_{n=0}^{\infty} c_n . x^{2n}$ *?*
3. *If the radius of convergence of the power series* $\sum_{n=0}^{\infty} c_n x^n$ *is 10, what is the radius of convergence of the series* $\sum_{n=1}^{\infty} nc_n x^{n-1}$ *?* $\sum_{n=1}^{\infty} \frac{c_n}{n+1} x^{n+1}$ *? Why ?*

3.3. Algebraic operations on convergent power series

Overview

Particular power series may be convergent for all values of x; or else, it may also not be convergent for any values of x except the one value x. However, its determination is not always easy.

In this section we study the convergence and divergence of power series, and also we will be able to determine whether or not a given power series is convergent or divergent by using properties.

Section objective:

After you complete the study of this subtopic, you should be able to:

- decide the convergence or divergence of a power series;

Lemma:

a. If $\sum_{n=0}^{\infty} c_n \, . s^n$ converges, then $\sum_{n=0}^{\infty} c_n \, . x^n$
   - Converges absolutely for $|x| < |s|$
   - Diverges when $|x| > |s|$

b. If $\sum_{n=0}^{\infty} c_n \, . s^n$ diverges , then $\sum_{n=0}^{\infty} c_n \, . x^n$
   - Converges absolutely for $|x| \leq |s|$
   - Diverges for $|x| > |s|$

Example 5:

1. Suppose that $\sum_{n=0}^{\infty} c_n \, . 3^n$ is convergent, then determine the convergence or divergence of the following power series.

a. $\sum_{n=0}^{\infty} c_n \, . 2^n$ b. $\sum_{n=0}^{\infty} c_n$ c. $\sum_{n=0}^{\infty} c_n \, . (-2)^n$ d. $\sum_{n=0}^{\infty} c_n \, . (-3)^n$

e. $\sum_{n=0}^{\infty} c_n \, . (-8)^n$ f. $\sum_{n=0}^{\infty} c_n \, . 5^n$.

Solution: Given $s = 3$

a. $x = 2 < 3$, hence converges.

b. $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} c_n \, . 1^n$ which implies $x = 1 < 3$, hence converges.

c. $\sum_{n=0}^{\infty} c_n \, . (-2)^n$ and $|x| = |-2| = 2 < 3$; thus, convergent.

d. $\sum_{n=0}^{\infty} c_n \,.\, (-3)^n \implies |-3| < 3$ which is false, therefore,divergent.

e. $\sum_{n=0}^{\infty} c_n \,.\, (-8)^n \implies |-8| = 8 > 3$ which is divergent.

f. $\sum_{n=0}^{\infty} c_n \,.\, 5^n \implies |5| > 3$ which is divergent.

2. Suppose that $\sum_{n=0}^{\infty} c_n \,.\, 3^n$ is divergent , then what can you say about the convergence or divergence of :

a. $\sum_{n=0}^{\infty} c_n \,.\, (-2)^n$ b. $\sum_{n=0}^{\infty} c_n \,.\, 5^n$ c. $\sum_{n=0}^{\infty} (-1)^n . c_n \,.\, 9^n$

d. $\sum_{n=0}^{\infty} (-1)^n . c_n \,.\, 3^n$

Solution:

a. $\sum_{n=0}^{\infty} c_n \,.\, (-2)^n$ and $|x| = |-2| = 2 < 3$, hence convergence.

b. $x = 5 > 3$, hence divergence.

c. $\sum_{n=0}^{\infty} (-1)^n . c_n \,.\, 9^n = \sum_{n=0}^{\infty} c_n \,.\, (-9)^n$ and $|x| = |-9| = 9 > 3$, hence diverges.

d. $\sum_{n=0}^{\infty} (-1)^n . c_n \,.\, 3^n = \sum_{n=0}^{\infty} c_n \,.\, (-3)^n$ and $|x| = |-3| = 3 > 3$ which is false and hence divergent.

Exercise 3.3

1. *If* $\sum_{n=0}^{\infty} c_n \,.\, 4^n$ *is convergent, then does it follow that the following series are convergent?*

i. $\sum_{n=0}^{\infty} c_n \,.\, (-2)^n$ *ii.* $\sum_{n=0}^{\infty} c_n \,.\, (-4)^n$

2. *Suppose that* $\sum_{n=0}^{\infty} c_n \,.\, x^n$ *converges when* $x = -4$ *and diverges when* $x = 6$ *. What can be said about the convergence or divergence of the following series ?*

a. $\sum_{n=0}^{\infty} c_n$ *b.* $\sum_{n=0}^{\infty} c_n \,.\, 8^n$ *c.* $\sum_{n=0}^{\infty} c_n \,.\, (-3)^n$

d. $\sum_{n=0}^{\infty} (-1)^n . c_n \,.\, 9^n$

3.4. Differentiation and integration of power series

Overview

A power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \cdots$ or $\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \ldots + a_n(x-a)^n + \cdots$ is a function, hence, its derivative and integration exists.

In this section we study the differentiation and integration of a power series one by one.

Section objective:

After you complete the study of this subtopic, you should be able to:

- Differentiate a power series term by term;
- Integrate a power series term by term;

3.4.1. Differentiation of Power Series

Since a power series is a function, then its derivative exists. A power series with a non-zero radius of convergence is always differentiable. This derivative is obtained from $\sum_{n=0}^{\infty} a_n x^n$ or $\sum_{n=0}^{\infty} a_n (x-a)^n$ by differentiating term by term inside the interval of convergence, the way we differentiate polynomials.

Theorem 2(Differentiation Theorem for Power Series)

If the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is differentiable, and

$$f'(x) = (c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots)' = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

Or,

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} .$$

Similarly, the derivative for a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with a radius of convergence $R > 0$ is given by:

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n\, x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}(a_n\, x^n) = \sum_{n=1}^{\infty} na_n\, x^{n-1}$$
$$= a_1 + 2a_2x + 3a_3x^3 + \cdots$$

for $|x| < R$.

Remark 5:

1. The initial index of the power series changes when we go from one derivative to another. For instance, for first derivative the initial index is 1 for $\frac{d}{dx}(\sum_{n=0}^{\infty} a_n x^n) = \sum_{n=1}^{\infty} na_n x^{n-1}$ and since the derivative of c_0x^0 is 0. For second derivative the initial index is 2 for $\frac{d}{dx}(\sum_{n=1}^{\infty} na_n x^{n-1}) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$ and so on.
2. The power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ and its derivative $\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ has the **<u>same radius of convergence</u>** but not necessarily the same interval of convergence.

Example 6: For each of the following power series, find $f'(x)$ and $f''(x)$.

a. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Solution: $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n!}\right) = \sum_{n=1}^{\infty} \frac{n}{n!}\, x^{n-1} = \sum_{n=1}^{\infty} \frac{n}{n.(n-1)!}\, x^{n-1}$$
$$= \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Also,

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n!}\right) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

b. $\sum_{n=0}^{\infty} (\frac{x}{5})^n$

Solution: $f(x) = \sum_{n=0}^{\infty} (\frac{x}{5})^n$

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} (\frac{x}{5})^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{x^n}{5^n}\right) = \sum_{n=1}^{\infty} \frac{n}{5^n}\, x^{n-1}$$

and,

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\sum_{n=1}^{\infty} \frac{n}{5^n} x^{n-1}\right) = \sum_{n=1}^{\infty} \frac{d}{dx}\left(\frac{n}{5^n}\, x^{n-1}\right) = \sum_{n=2}^{\infty} \frac{n(n-1)}{5^n}\, x^{n-2} .$$

c. $\sum_{n=1}^{\infty}\frac{(-1)^n.(x+2)^n}{n.2^n}$

Solution: $f(x) = \sum_{n=1}^{\infty}\frac{(-1)^n.(x+2)^n}{n.2^n}$

$$f'(x) = \frac{d}{dx}\left(\sum_{n=1}^{\infty}\frac{(-1)^n.(x+2)^n}{n.2^n}\right) = \sum_{n=0}^{\infty}\frac{d}{dx}\left(\frac{(-1)^n.(x+2)^n}{n.2^n}\right) = \sum_{n=1}^{\infty}\frac{(-1)^n.n.(x+2)^{n-1}}{n.2^n}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n.\ (x+2)^{n-1}}{2^n}$$

and,

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\sum_{n=1}^{\infty}\frac{(-1)^n.\ (x+2)^{n-1}}{2^n}\right) = \sum_{n=1}^{\infty}\frac{d}{dx}\left(\frac{(-1)^n.\ (x+2)^{n-1}}{2^n}\right)$$

$$= \sum_{n=2}^{\infty}\frac{(-1)^n.(n-1).\ (x+2)^{n-2}}{2^n}.$$

3.4.2. Integration of Power Series

The power series can also be integrated term-by-term on an interval lying inside the interval of convergence.

Theorem 3(Integration Theorem for a Power Series)

If the power series $\sum_{n=0}^{\infty}c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = \sum_{n=0}^{\infty}c_n\,(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is integrable, and

$$\int f(x)dx = \int\sum_{n=0}^{\infty}c_n\,(x-a)^n\,dx = \sum_{n=0}^{\infty}\int c_n\,(x-a)^n dx = \sum_{n=0}^{\infty}\frac{c_n}{n+1}(x-a)^{n+1} + k$$

where k is a constant.

Similarly, if a power series $f(x) = \sum_{n=0}^{\infty}c_n\,x^n$ has a radius of convergence $R > 0$, then it is integrable, and it is given by:

$$\int f(x)dx = \int\sum_{n=0}^{\infty}c_n\,x^n\,dx = \sum_{n=0}^{\infty}\int c_n\,x^n dx = \sum_{n=0}^{\infty}\frac{c_n}{n+1}x^{n+1} + k$$

Remark 6:

1. The power series $\sum_{n=0}^{\infty}c_n\,(x-a)^n$ and its integration $\sum_{n=0}^{\infty}\frac{c_n}{n+1}\,(x-a)^{n+1}$ has the **same radius of convergence** but not for interval of convergence.
2. Unlike to differentiation, the initial index of the power series do not change even after integration is commenced.

Example 7: For each of the following power series, find $\int f(x)dx$.

a. $\sum_{n=1}^{\infty} \frac{n+1}{n}(x+2)^n$

Solution: $f(x) = \sum_{n=1}^{\infty} \frac{n+1}{n}(x+2)^n$

$$\int f(x)dx = \int \sum_{n=1}^{\infty} \frac{n+1}{n}(x+2)^n\, dx = \sum_{n=1}^{\infty} \int \frac{n+1}{n}(x+2)^n dx = \sum_{n=1}^{\infty} \frac{(x+2)^n}{n} + k$$

b. $\sum_{n=1}^{\infty} \frac{(x-1)^{n+1}}{n+1}$

Solution: $f(x) = \sum_{n=1}^{\infty} \frac{(x-1)^{n+1}}{n+1}$

$$\Rightarrow \int f(x)dx = \int \sum_{n=1}^{\infty} \frac{(x-1)^{n+1}}{n+1}\, dx = \sum_{n=1}^{\infty} \int \frac{(x-1)^{n+1}}{n+1} dx == \sum_{n=1}^{\infty} \frac{(x-1)^{n+2}}{(n+1)(n+1)} + k$$

$$= \sum_{n=1}^{\infty} \frac{(x-1)^{n+2}}{n^2+3n+2} + k$$

Example 8: For each of the following power series, find $f'(x)$, $f''(x)$ and $\int f(x)dx$

a. $\sum_{n=1}^{\infty}(n+1)x^n$

Solution: $f(x) = \sum_{n=1}^{\infty}(n+1)x^n$

$$f'(x) = \frac{d}{dx}\left(\sum_{n=1}^{\infty}(n+1)x^n\right) = \sum_{n=1}^{\infty} \frac{d}{dx}\left((n+1)x^n\right) = \sum_{n=1}^{\infty} n(n+1)x^{n-1}$$

and,

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\sum_{n=1}^{\infty} n(n+1)x^{n-1}\right) = \sum_{n=1}^{\infty} \frac{d}{dx}\left(n(n+1)x^{n-1}\right)$$

$$= \sum_{n=2}^{\infty} n(n+1)(n-1)x^{n-2}$$

Also,

$$\int f(x)dx = \int \sum_{n=1}^{\infty}(n+1)x^n\, dx = \sum_{n=1}^{\infty} \int (n+1)x^n dx = \sum_{n=1}^{\infty} x^n + k$$

b. $\sum_{n=1}^{\infty} \frac{5}{n}.x^{n^2}$

Solution: $f(x) = \sum_{n=1}^{\infty} \frac{5}{n}.x^{n^2}$

$$f'(x) = \frac{d}{dx}\left(\sum_{n=1}^{\infty} \frac{5}{n}x^{n^2}\right) = \sum_{n=1}^{\infty} \frac{d}{dx}\left(\frac{5}{n}x^{n^2}\right) = \sum_{n=1}^{\infty} \frac{5}{n}.n^2.x^{n^2-1} = \sum_{n=1}^{\infty} 5n.x^{n^2-1}$$

and,

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\sum_{n=1}^{\infty} 5n.x^{n^2-1}\right) = \sum_{n=1}^{\infty} \frac{d}{dx}\left(5n.x^{n^2-1}\right)$$

$$= \sum_{n=2}^{\infty} 5n.(n^2-1).x^{n^2-2}$$

Also,

$$\int f(x)dx = \int \sum_{n=1}^{\infty} \frac{5}{n}.x^{n^2}\ dx = \sum_{n=1}^{\infty} \int \frac{5}{n}.x^{n^2} dx \quad = \sum_{n=1}^{\infty} \frac{5}{n(n^2+1)}.x^{n^2+1}\ + k$$

c. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.x^n$

Solution: $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.x^n$

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}x^n\right) = \sum_{n=1}^{\infty} \frac{d}{dx}\left(\frac{(-1)^n}{n+1}x^n\right) = \sum_{n=1}^{\infty} \frac{(-1)^n.\ n}{n+1}.x^{n-1}$$

$$= \sum_{n=1}^{\infty}(-1)^n.\frac{n}{n+1}.x^{n-1}$$

and,

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\sum_{n=1}^{\infty}(-1)^n.\frac{n}{n+1}.x^{n-1}\right) = \sum_{n=1}^{\infty} \frac{d}{dx}\left((-1)^n.\frac{n}{n+1}.x^{n-1}\right)$$

$$= \sum_{n=2}^{\infty}(-1)^n.\frac{n(n-1)}{n+1}.x^{n-2}$$

Also,

$$\int f(x)dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}.x^n\ dx = \sum_{n=1}^{\infty} \int \frac{(-1)^n}{n+1}.x^n dx \quad = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^2}.x^{n+1}\ + k$$

d. $\sum_{n=0}^{\infty} \frac{1}{n^2+1}.x^{n+1}$

Solution: $f(x) = \sum_{n=0}^{\infty} \frac{1}{n^2+1}.x^{n+1}$

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{1}{n^2+1}.x^{n+1}\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{1}{n^2+1}.x^{n+1}\right) = \sum_{n=0}^{\infty} \frac{n+1}{n^2+1}.x^n$$

and,

$$f''(x) = \frac{d}{dx}(f'(x)) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{n+1}{n^2+1}.x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}\left(\frac{n+1}{n^2+1}.x^n\right)$$

$$= \sum_{n=1}^{\infty} \frac{n(n+1)}{n^2+1}.x^{n-1}$$

Also,

$$\int f(x)dx = \int \sum_{n=0}^{\infty} \frac{1}{n^2+1}x^{n+1}dx = \sum_{n=1}^{\infty} \int \frac{1}{n^2+1}x^{n+1}dx = \sum_{n=1}^{\infty} \frac{1}{(n+2)(n^2+1)}x^{n+2}\ + k$$

Example 9:

a. Find a power series representation for the function $\ln(1+x), \quad |x| < 1.$

Solution:

From example 2, (3.1), we found the power series expansion

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty}(-1)^n x^n \quad , \quad |x| < 1\ .$

Integrating this series term-by-term on the interval [0; x], we find that

$\ln(1+x) =$

$$\int_0^x \frac{dx}{1+t} = \int_0^x [1 - t + t^2 - t^3 + \cdots]dt = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^n}{n} \ ,$$

b. Represent the integral $\int_0^x \frac{\ln(1+t)}{t} dt$ as a power series expansion.

Solution.

In the previous problem (Example 1) we have found the power series expansion for logarithmic function:

$$\ln(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}t^n}{n} = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots, \qquad |t| < 1 \ .$$

Then we can write:

$$\frac{\ln(1+t)}{t} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}t^{n-1}}{n} = 1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \frac{t^4}{5} - \frac{t^5}{6} + \cdots, \qquad |t| < 1$$

Integrating this series term-by-term on the interval [0; x], we obtain

$$\int_0^x \frac{\ln(1+t)}{t} dt = \int_0^x \left[1 - \frac{t}{2} + \frac{t^2}{3} - \frac{t^3}{4} + \right] dt = x - \frac{x^2}{2.2} + \frac{x^3}{3.3} - \frac{x^4}{4.4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n^2}$$

c. Obtain a power series representation for the exponential function e^x.

Solution.

Consider the series $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ that converges for all x. Differentiating it term-by-term, we have

$$f'(x) = \frac{d}{dx}(1) + \frac{d}{dx} x + \frac{d}{dx}\frac{x^2}{2!} + \frac{d}{dx}\frac{x^3}{3!} + \cdots = 0 + 1 + x + \frac{x^2}{2!} + \cdots = f(x)$$

Hence, the function f (x) satisfies the differential equation f ' = f. The general solution of this equation has the form f (x) = ce ^{x}, where c is a constant.

Substituting the initial value f (0) = 1, we find that c = 1.

Thus, we obtain the following power series expansion for e ^{x}:

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

d. Find a power series expansion for the hyperbolic sine function sinh x.

Solution.

Since sinh x = (e ^{x} + e $^{-x}$)/2, we can use power series representations for e ^{x} and e $^{-x}$. In the previous example we found the formula

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Substituting $-x$ for x, we get

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

Then the expansion for the hyperbolic sine function has the form:

$\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{x^n}{n!} - \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!}\right] = \frac{1}{2}[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - (1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots)]$

$$= \frac{1}{2}\left[2\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)\right] = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

Exercise 3.4

1. *For each of the following power series, find* $f'(x)$, $f''(x)$ *and* $\int f(x)dx$.

   a. $f(x) = \sum_{n=0}^{\infty} 10^n x^n$ b. $f(x) = \sum_{n=1}^{\infty}\frac{n+1}{n}.x^n$ c. $f(x) = \sum_{n=0}^{\infty} x^{2n}$

   d. $f(x) = \sum_{n=1}^{\infty} n^{-n} x^n$ e. $f(x) = \sum_{n=1}^{\infty}\sqrt{n}.\sqrt{n+1}.x^n$ f. $f(x) = \sum_{n=1}^{\infty}\frac{1}{n^2}.x^{2n}$

2. *If the radius of convergence of the power series* $\sum_{n=0}^{\infty} c_n .x^n$ *is 10, what is the radius of convergence of the series* $\sum_{n=1}^{\infty} n.c_n .x^{n-1}$? $\sum_{n=1}^{\infty}\frac{c_n}{n+1}.x^{n+1}$? *Why ?*

3.5. Taylor series and Maclaurin series

Overview

In this section we will discuss methods for finding power series for derivatives and integrals of functions, and we will also discuss practical methods for finding Taylor series that can be used in situations where it is difficult or impossible to find the series directly.

Section objective:

After you complete the study of this subtopic, you should be able to:

- distinguish Taylor and Maclaurin series;
- find Taylor series of a function;
- find Maclaurin series of a function.

When a function is written in the form an infinite series, it is said to be ‘expanded’ in an infinite series. This series represents all values of x in the interval of convergence.

For the function, the infinite series is:

$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$

or,

$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 + \cdots$

The function f(x) has the following properties of a polynomial.

- It is continuous with the interval of convergence (there is no break in its graph).
- In series form, the function can be added, subtracted, multiplied or divided term by term.
- If f(x) is differentiable, then the series can be differentiated term by term.

Two common series representing expansions are the Maclaurin series and the Taylor series. In these series, successive derivatives are taken and the coefficients can be obtained

If a function f (x) has continuous derivatives up to $(n+1)$th order, then this function can be expanded in the following form:

$$f(x) = \sum_{n=0}^{\infty} f^n(a)\frac{(x-a)^n}{n!} = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^n(a)(x-a)^n}{n!} + R_n$$

where R_n , the remainder after n+1 terms, is given by $R_n = \frac{f^{n+1}(\xi)(x-a)^{n+1}}{(n+1)!}$, $a < \xi < x$.

- When this expansion converges over a certain range of x, that is, $\lim_{n\to\infty} R_n = 0$, then the expansion is called **Taylor series** of f (x) about a, that is,

  $$f(x) = \sum_{n=0}^{\infty} f^n(a)\frac{(x-a)^n}{n!} = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^n(a)(x-a)^n}{n!}$$

  is called Taylor series of f (x) about a.
- If $a = 0$, the Taylor series is called Maclaurin series, that is,

  $$f(x) = \sum_{n=0}^{\infty} f^n(0)\frac{x^n}{n!} = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots + \frac{f^n(0)x^n}{n!}$$

  is called Maclaurin series of f(x) or Taylor series of f (x) about $a = 0$.

Theorem 4: If f has a power series representation at a , that is, if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$, $|x-a| < R$, then its coefficients are given by $c_n = \frac{f^n(a)}{n!}$.

Proof: Suppose f is any function that can be represented by a power series.

1) $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$

$\Rightarrow f(a) = c_0 = 0!\, c_0$

2) Take the first derivative of each term:

$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \dots$

$\Rightarrow f'(a) = c_1 = 1!\, c_1$

3) Take the second derivative of each term:

$f''(x) = \sum_{n=2}^{\infty} n.(n-1).c_n.(x-a)^{n-2} = 2c_2 + 2.3c_3(x-a) + 3.4.c_4(x-a)^2 + 4.5.c_5(x-a)^3 + \dots$

$\Rightarrow f''(a) = 2c_2 = 2!\, c_2$

4) Take the third derivative of each term:

$f'''(x) = \sum_{n=3}^{\infty} n.(n-1).(n-2).c_n.(x-a)^{n-3} = 2.3c_3 + 2.3.4c_4(x-a) + 3.4.5.c_5(x-a)^2 + 4.5.6.c_6(x-a)^3 + \dots$

$\Rightarrow f'''(a) = 2.3c_3 = 3!\, c_3$

5) Take the 4^{th} derivative of each term:

$f^4(x) = \sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)c_n\,(x-a)^{n-4}$

$= 2.3.4.c_4 + 2.3.4.5.c_5(x-a) + 3.4.5.6.c_6(x-a)^2 + 4.5.6.7.c_7(x-a)^3 + \dots$

$\Rightarrow f^4(a) = 2.3.4.c_4 = 4!\, c_4$

If we continue to differentiate, we get

$$f^n(x) = n!\,c_n + (n+1)!\,c_{n+1}x + (n+2)!\,c_{n+2}x^2 \dots$$

From this, we obtain:

$$f^n(a) = 2.3.4 \dots n.c_n = n!\,c_n$$

Solving for c_n, we obtain $c_n = \frac{f^n(a)}{n!}$.

Definition 5(Taylor series): If f has a power series representation at a , then any expression of the form: $f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}.(x-a)^n = f(a) + \frac{f'(a)}{1!}.(x-a) + \frac{f''(a)}{2!}.(x-a)^2 + \frac{f'''(a)}{3!}.(x-a)^3 + \cdots$ is called the Taylor series of the function f at a .

If $a = 0$, then $f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}.x^n = f(0) + \frac{f'(0)}{1!}.x + \frac{f''(0)}{2!}.x^2 + \frac{f'''(0)}{3!}.x^3 + \cdots$ is called the Maclaurin series.

Remark 7:

1. If a function has Taylor series, then the function must be infinitely differentiable. But not the converse, there are functions which are infinitely differentiable but does not have Taylor series.
2. To represent f(x) as a sum of power series in $(x-a)$ or Taylor Series in $(x-a)$, we need to consider the following steps:

   Step1: Compute all the derivatives $f^{n}(a)$, $n = 0,1,2,3,4,\dots$ where $f^{0}(a) = f(a)$. If these derivatives does not all exist, f(x) is not the sum of a power series in a power of $(x-a)$.

   Step2: Write down the Taylor series of f(x) at $x = a$.

Example 10:

1. Given the function $f(x) = e^{x}$, then find the
   i. Taylor series at $a = 2$.
   ii. Maclaurin series.

Solution:

i. We have $f(x) = e^{x}$ and by definition the Taylor series at a is $f(x) = \sum_{n=0}^{\infty}\frac{f^{n}(a)}{n!}(x-a)^{n}$.

Step1: Compute all the derivatives $f^{n}(a)$, $n = 0,1,2,3,\dots$

$f'(x) = f''(x) = \cdots = f^{n}(x) = e^{x}$

$\Longrightarrow f^{n}(a) = f^{n}(2) = e^{2}$ for $n = 0,1,2,3,\dots$

Step2: Write down the Taylor series of f(x) at $x = a$.

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{n}(a)}{n!}(x-a)^{n}$$

$$= f(a) + f'(a).(x-a) + \frac{f''(a)}{2!}.(x-a)^{2} + \frac{f'''(a)}{3!}.(x-a)^{3} + \cdots$$

$$= f(2) + f'(2).(x-2) + \frac{f''(2)}{2!}.(x-2)^{2} + \frac{f'''(2)}{3!}.(x-2)^{3} + \dots$$

$$= e^{2} + e^{2}.(x-2) + \frac{e^{2}}{2!}.(x-2)^{2} + \frac{e^{2}}{3!}.(x-2)^{3} + \cdots \text{ or } = \sum_{n=0}^{\infty}\frac{e^{2}}{n!}.(x-2)^{n}$$

Therefore, the Taylor series $f(x) = e^{x}$ at $x = 2$ is $e^{x} = \sum_{n=0}^{\infty}\frac{e^{2}}{n!}.(x-2)^{n}$ for all x .

ii. Given $f(x) = e^{x}$, then $f^{n}(x) = e^{x} \Longrightarrow f^{n}(0) = e^{0} = 1$ for all n .

Therefore, the Taylor series representation of f at $a = 0$ (called Maclaurin series) is

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} \cdot x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ \dots$$

$$\Longrightarrow \ e^x \ = \ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ \dots$$

Remark 8: We have two power series representation for e^x , that is, the Maclaurin series and the Taylor series at $a = 2$. This implies Taylor series is not unique and it varies with the centre a.

2. Find the Maclaurin series and Taylor series of the following functions

i. $f(x) = \sin x$ at $a = \frac{\pi}{2}$

Solution:

Step1: Compute all the derivative, i.e., $f^n(0)$, $n = 0, 1, 2, 3, 4, \dots$

$$f(x) = \sin x \quad \Longrightarrow f(0) = 0 ,$$

$$f'(x) = \cos x \quad \Longrightarrow f'(0) = 1 ,$$

$$f''(x) = -\sin x \quad \Longrightarrow f''(0) = 0 ,$$

$$f'''(x) = -\cos x \quad \Longrightarrow f'''(0) = -1 .$$

Again this numbers will be repeated,

$$f^4(x) = \sin x \quad \Longrightarrow f^4(0) = 0$$

$$f^5(x) = \cos x \quad \Longrightarrow f^5(0) = 1$$

$$f^6(x) = -\sin x \quad \Longrightarrow f^6(0) = 0$$

$$f^7(x) = -\cos x \quad \Longrightarrow f^7(0) = -1$$

Since the derivative repeat in a cycle of four,

a) The Maclaurin series is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n = f(0) + f'(0)x + \frac{f^2(0)}{2!} x^2 + \frac{f^3(0)}{3!} x^3 + \cdots$$

$$= 0 + x + 0 - \frac{x^3}{3!} + 0 - \frac{x^5}{5!} + \ \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$$

b) The Taylor series at $a = \frac{\pi}{2}$

Step1: Compute all the derivatives at a i.e. $f^n(a)$, $n = 0, 1, 2, 3, \ldots$

$f\left(\frac{\pi}{2}\right) = \sin(\frac{\pi}{2}) = 1$

$f'\left(\frac{\pi}{2}\right) = \cos(\frac{\pi}{2}) = 0$

$f''\left(\frac{\pi}{2}\right) = -\sin(\frac{\pi}{2}) = -1$

$f'''\left(\frac{\pi}{2}\right) = -\cos(\frac{\pi}{2}) = 0$

Since the derivatives repeat in a cycle of four, the respective coefficients for 4^{th}, 5^{th}, 6^{th} and 7^{th} derivatives are 1, 0, -1, 0.

The Taylor series at a is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} .(x-a)^n = f(a) + f'(a).(x-a) + \frac{f^2(a)}{2!}.(x-a)^2 + \frac{f^3(a)}{3!}.(x-a)^3 + \cdots$$

Since $a = \frac{\pi}{2}$, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(\frac{\pi}{2})}{n!} .(x-\frac{\pi}{2})^n$$

$$= f\left(\frac{\pi}{2}\right) + \frac{f'\left(\frac{\pi}{2}\right)}{1!}\left(x-\frac{\pi}{2}\right) + \frac{f''\left(\frac{\pi}{2}\right)}{2!}.(x-\frac{\pi}{2})^2 + \frac{f^3\left(\frac{\pi}{2}\right)}{3!}.(x-\frac{\pi}{2})^3 + \ldots$$

$$= 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} - \frac{(x-\frac{\pi}{2})^6}{6!} + \frac{(x-\frac{\pi}{2})^8}{8!} \ldots$$

$$= \sum_{n=0}^{\infty}(-1)^n . \frac{(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$$

ii. $\cos x$

Solution:

Step1: Compute all the derivative $f^n(0), n = 0, 1, 2, 3, 4, \ldots$

$f(x) = \cos x \implies f(0) = 1$

$f'(x) = -\sin x \implies f'(0) = 0$

$f''(x) = -\cos x \implies f''(0) = -1$

$f'''(x) = \sin x \implies f'''(0) = 0$

Again this numbers will be repeated

$f^4(x) = \cos x \implies f^4(0) = 1$

$f^5(x) = -\sin x \implies f^5(0) = 0$

$f^6(x) = -\cos x \implies f^6(0) = -1$

$f^7(x) = \sin x \implies f^7(0) = 0$

Since the derivative repeat in a cycle of four, the Maclaurin series is given by

$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} . x^n = f(0) + f'(0). x + \frac{f^2(0)}{2!}. x^2 + \frac{f^3(0)}{3!}. x^3 + \cdots$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

$= \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n}}{(2n)!}$

Or we know that $\cos x = \frac{d}{dx}(\sin x)$ and $\sin x = \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n+1}}{(2n+1)!}$

$\implies \frac{d}{dx} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right)$

$\implies 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6\,!} + \cdots$

$= \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n}}{(2n)!} = \cos x$

a. $f(x) = x . \cos x$

Solution: So far we know the Maclaurin series for $\cos x$ which is given by

$f(x) = \cos x = \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n}}{(2n)!}$

Then $x . \cos x = x. \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n}}{(2n)!}$

$= \sum_{n=0}^{\infty} (-1)^n . \frac{x . x^{2n}}{(2n)!}$

$= \sum_{n=0}^{\infty} (-1)^n . \frac{x^{2n+1}}{(2n)!}$

b. $f(x) = \frac{1}{1-x}$

Solution:

Step1: Compute all the derivative $f^n(0), n = 0, 1, 2, 3, 4, \ldots$

$f(x) = \frac{1}{1-x} \implies f(0) = 1$

$f'(x) = \frac{1}{(1-x)^2} \implies f'(0) = 1$

$f''(x) = \frac{2}{(1-x)^3} \implies f''(0) = 2$

$f'''(x) = \frac{6}{(1-x)^4} \implies f'''(0) = 6$

$f^4(x) = \frac{24}{(1-x)^5} \implies f^4(0) = 24$

$f^5(x) = \frac{120}{(1-x)^6} \Rightarrow f^5(0) = 120$

Step2: Write out the Maclaurin series

$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}.x^n = f(0) + f'(0).x + \frac{f^2(0)}{2!}.x^2 + \frac{f^3(0)}{3!}.x^3 + \cdots$

$= 1 + x + \frac{2x^2}{2!} + \frac{6x^3}{3!} + \frac{24x^4}{4!} + \frac{120x^5}{5!} + \cdots$

$= 1 + x + x^2 + x^3 + x^4 + \cdots$

$= \sum_{n=0}^{\infty} x^n$

Thus $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

iii. $f(x) = \frac{1}{1+x}$

Solution:

Step1: Compute all the derivative $f^n(0), n = 1, 2, 3, 4, \ldots$

$f(x) = \frac{1}{1+x} \Rightarrow f(0) = 1$

$f'(x) = \frac{-1}{(1+x)^2} \Rightarrow f'(0) = -1$

$f''(x) = \frac{2}{(1+x)^3} \Rightarrow f''(0) = 2$

$f'''(x) = \frac{-6}{(1+x)^4} \Rightarrow f'''(0) = -6$

$f^4(x) = \frac{24}{(1+x)^5} \Rightarrow f^4(0) = 24$

$f^5(x) = \frac{-120}{(1+x)^6} \Rightarrow f^5(0) = -120$

Step2: Write out the Maclaurin series

$f(x) = \frac{1}{1+x} = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}.x^n = f(0) + f'(0).x + \frac{f^2(0)}{2!}.x^2 + \frac{f^3(0)}{3!}.x^3 + \cdots$

$= 1 - x + \frac{2x^2}{2!} - \frac{6x^3}{3!} + \frac{24x^4}{4!} - \frac{120x^5}{5!} + \cdots$

$= 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$

$= \sum_{n=0}^{\infty} x^n$

Thus $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n.\ x^n$

Or,

$f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$

Remark 9: Taylor series enables us to integrate functions that we could not integrate

Example 11: Evaluate $\int e^{-x^2} dx$

Solution: It is impossible to integrate $\int e^{-x^2} dx$ by any one of the methods. Thus we first find the Maclaurin series for $f(x) = e^{-x^2}$ by any of the methods we have. Hence we need to device another mechanism by which we can integrate this and such functions and for this we have the Maclaurin series. Now we first find the Maclaurin series for the function $f(x) = e^{-x^2}$.

$$\Longrightarrow\ e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n . x^{2n}}{n!} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots$$

Now integrate term by term

$$\int e^{-x^2} dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n . x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \int \frac{(-1)^n . x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n . x^{2n+1}}{n! . (2n+1)}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \ \ldots$$

Example 12: find the Taylor series for $f(x)$ centered at the given values of a .(Assume that f has a power series representation)

a. $f(x) = 1 + x + x^2$, $a = 2$

Solution: To find the Taylor series of $f(x) = 1 + x + x^2$

Step1: Compute all the derivatives $f^n(a)$, $n = 0 , 1 , 2 , 3$

$f(x) = 1 + x + x^2 \Longrightarrow f(2) = 7$

$\Longrightarrow f'(x) = 1 + 2x \Longrightarrow f'(2) = 5$

$\Longrightarrow f''(x) = 2 \Longrightarrow f''(2) = 2$

$f^n(x) = 0$ when $n \geq 3 \Longrightarrow f^n(2) = 0$ when $n \geq 3$

Step2: write down the Taylor series of $f(x)$ at $a = 2$

$$\Longrightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^n(2)}{n!} . (x-2)^n$$

$$= f(2) + f'(2). (x-2) + \frac{f''(2)}{2!} . (x-2)^2 + \frac{f'''(2)}{3!} . (x-2)^3 + \cdots$$

$$= 7 + 5. (x-2) + \frac{2}{2!} . (x-2)^2 + \ \ldots$$

$$= 7 + 5(x-2) + (x-2)^2$$

b. $f(x) = x^3$, $a = -1$

Solution: To find the Taylor series of $f(x) = x^3$

Step1: Compute all the derivatives $f^n(a)$, $n = 0, 1, 2, 3$

$f(x) = x^3 \implies f(-1) = -1$

$\implies f'(x) = 3x^2 \implies f'(-1) = 3$

$\implies f''(x) = 6x \implies f''(-1) = -6$

$\implies f'''(x) = 6 \implies f'''(-1) = 6$

$f^n(x) = 0$ when $n \geq 4 \implies f^n(-1) = 0$ when $n \geq 4$

Step2: write down the Taylor series of $f(x)$ at $a = -1$

$\implies f(x) = \sum_{n=0}^{\infty} \frac{f^n(-1)}{n!} . (x+1)^n$

$= f(-1) + f'(-1). (x+1) + \frac{f''(-1)}{2!} . (x+1)^2 + \frac{f'''(-1)}{3!} . (x+1)^3 + \cdots$

$= -1 + 3. (x+1) + 3(x+2)^2 + (x+1)^3$

c. $f(x) = e^x$, $a = 3$

Solution:

We have $f(x) = e^x$ and by definition the Taylor series at $a = 3$ is $f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} . (x-a)^n$

Step 1: Compute all the derivatives $f^n(a)$, $n = 0, 1, 2, 3, \ldots$

$\implies f'(x) = f''(x) = \cdots = f^n(x) = e^x$

$\implies f^n(a) = f^n(3) = e^3$

Step2: write down the Taylor series of $f(x)$ at $a = 3$

$\implies f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} . (x-a)^n$

$= f(a) + f'(a). (x-a) + \frac{f''(a)}{2!} . (x-a)^2 + \frac{f'''(a)}{3!} . (x-a)^3 + \cdots$

$= f(3) + f'(3). (x-3) + \frac{f''(3)}{2!} . (x-3)^2 + \frac{f'''(3)}{3!} . (x-3)^3 + \ldots$

$= e^3 + e^3. (x-3) + \frac{e^3}{2!} . (x-3)^2 + \frac{e^3}{3!} . (x-3)^3 + \cdots$

$= \sum_{n=0}^{\infty} \frac{e^3}{n!} . (x-3)^n$

$\implies e^x = \sum_{n=0}^{\infty} \frac{e^3}{n!} . (x-3)^n$ for all x

Example 13:

a) Find the Maclaurin series for $\cos^2 x$.

Solution.

We use the trigonometric identity $\cos^2 x = \frac{1+\cos 2x}{2}$.

Since the Maclaurin series for cos x is $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, we can write:

$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$

Therefore

$1 + \cos 2x = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} = 2 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$

$\cos^2 x = \frac{1+\cos 2x}{2} = \frac{2+\sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}}{2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}$

b) Obtain the Taylor series for $f(x) = 3x^2 - 6x + 5$ around the point x = 1.

Solution.

Compute the derivatives:

$f'(x) = 6x - 6$, $f''(x) = 6$, $f'''(x) = 0$

As can be seen $f^n(x) = 0$ for all $n \geq 3$. Then, for x = 1, we get

$f(1) = 2$, $f'(1) = 0$, $f''(x) = 6$.

Hence, the Taylor expansion for the given function is

$f(x) = \sum_{n=0}^{\infty} f^n(1) \frac{(x-1)^n}{n!} = 2 + \frac{6(x-1)^2}{2!} = 2 + 3(x-1)^2$

c) Find the Maclaurin series for e^{kx}, k is a real number.

Solution.

Calculate the derivatives:

$f'(x) = (e^{kx})' = ke^{kx}$, $f''(x) = (ke^{kx})' = k^2 e^{kx}$, ... $f^n(x) = k^n e^{kx}$.

Then, at x = 0 we have

$f(0) = e^0 = 1$, $f'(0) = ke^x = k$, $f''(0) = k^2 e^0 = k^2$, $f^n(0) = k^n e^0 = k^n$

Hence, the Maclaurin expansion for the given function is

$e^{kx} = \sum_{n=0}^{\infty} f^n(0) \frac{x^n}{n!} = 1 + kx + \frac{k^2x^2}{2!} + \frac{k^3x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{k^n x^n}{n!}$

d. Find the Taylor series of the cubic function x^3 about x = 2.

Solution:

Denote $f(x) = x^3$. Then

$f'(x) = (x^3)' = 3x^2$, $f''(x) = (3x^2)'' = 6x$, $f'''(x) = (6x)' = 6$, $f^4(x) = 0$, and further $f^n(x) = 0$ for all $n \geq 4$.

Respectively, at the point $x = 2$, we have

$f(2) = 8, \quad f'(2) = 12, \quad f''(2) = 12, \quad f'''(2) = 6$.

Hence, the Taylor series expansion for the cubic function is given by the expression

$x^3 = \sum_{n=0}^{\infty} f^n(2)\frac{(x-2)^n}{n!} = 8 + 12(x-2) + \frac{12(x-2)^2}{2!} + \frac{6(x-2)^3}{3!} = 8 + 12(x-2) + 6(x-2)^2 + (x-2)^3$

e. Determine the Maclaurin series for $f(x) = \sqrt{1+x}$.

Solution.

Using the binomial series found in the previous example and substituting $\mu = \frac{1}{2}$, we get

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{x}{2} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)(\frac{1}{2}-2)}{3!}x^3 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{1.x^2}{2^2 2!} + \frac{1.3.x^3}{2^3 3!} - \frac{1.3.5.x^4}{2^4 4!} + \cdots + (-1)^{n+1}\frac{1.3.5.(2n-3)x^n}{2^n n!}.$$

Keeping only the first three terms, we can write this series as $\sqrt{1+x} \approx 1 + \frac{x}{2} - \frac{x^2}{8}$

Exercise 3.5

1. *If* $f(x) = \sum_{n=0}^{\infty} c_n \cdot (x-2)^n$ *for all* n*, then what is the formula for*
   a. c_5 *b.* c_9 *c.* c_n
2. *If* $f^n(0) = (n+1)!$ *for all* $n = 1,2,3, \ldots$ *, then find the Maclaurin series for* f *and its radius of convergence?*
3. *Find the Taylor series for* f *centered at 4 if* $f^n(4) = \frac{(-1)^n \cdot n!}{3^n \cdot (n+1)}$ *. What is the radius of convergence of the Taylor series ?*
4. *Find the Taylor series for* $f(x)$ *centered at the given values of* a
   a. $f(x) = 5 + 2x + 3x^2 + x^3 + x^4$; $a=2$
   b. $f(x) = \cos x$; $a=\pi$
   c. $f(x) = \ln x$; $a=2$
   d. $f(x) = \sin x$; $a={}^{\pi}/_{2}$
   e. $f(x) = x^{-2}$; $a=1$

3.6. Taylor polynomial and its application

Overview

In this section we will discuss methods for finding Taylor polynomial and use this to approximate Taylor polynomial function

Section objective:

After you complete the study of this subtopic, you should be able to:

- define Taylor polynomial;
- find Taylor polynomial;
- use Taylor Polynomial to approximate a function.

Taylor polynomials are applicable in approximating functions because polynomials are the simplest of functions.

Definition 6: Suppose that $f(x)$ is equal to the sum of its Taylor series at a, that is,

$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}.(x-a)^n$, then its n^{th} - partial sum of the Taylor series denoted by $P_n(x)$ which is called the n^{th} degree Taylor polynomial of f at a is given by

$$p_n(x) = \sum_{i=0}^{n} \frac{f^i(a)}{i!}.(x-a)^i = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}.(x-a)^n$$

Thus $p_n(x)$ can be used as an approximation to f i.e. $f(x) \cong p_n(x)$.

Remark 10:

1. $p_n(x) = \sum_{i=0}^{n} \frac{f^i(a)}{i!}.(x-a)^i$

   When $n = 1$, $p_1(x) = f(a) + f'(a).(x-a)$

   When $n = 2$, $p_2(x) = f(a) + f'(a).(x-a) + \frac{f''(a)}{2!}(x-a)^2$

2. When using a Taylor polynomial p_n to approximate a function f , we have to ask the question how good an approximation is it . To answer this we need to look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

   As $n \to \infty$, $|R_n(x)| \to \infty$ and $p_n(x) \to f(x)$.

Example 14:

1. Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$

Solution: $f(x) = \sqrt[3]{x} = x^{1/3}$

Step1: Compute the derivative $f^n(a)$ until $n = 2$

$f(x) = x^{1/3} \Rightarrow f(8) = 8^{1/3} = 2$

$f'(x) = \frac{1}{3x^{2/3}} \Rightarrow f'(8) = \frac{1}{3.(8^{2/3})} = \frac{1}{12}$

$f''(x) = \frac{-2}{9x^{5/3}} \Rightarrow f''(8) = \frac{-2}{9.(8^{5/3})} = \frac{-1}{144}$

Thus the second degree Taylor polynomial is

$$p_2(x) = \sum_{i=0}^{2} \frac{f^i(8)}{i!}.(x-8)^i = f(8) + f'(8)(x-8) + \frac{f''(8)}{2!}(x-8)^2$$

$$= 2 + \frac{1}{12}(x-8) - \frac{1}{144}.\frac{1}{2!}(x-8)^2$$

$$= 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

Thus the desired approximation is $\sqrt[3]{x} \cong p_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$

2. Find the Taylor polynomial $p_n(x)$ for the function f at the number a .

a. $f(x) = \sin x$ at $a = \frac{\pi}{6}$, $n = 3$

Solution:

Compute all the derivative $f^n(0)$, $n = 1, 2, 3$

$f(x) = \sin x \Rightarrow f\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$f'(x) = \cos x \Rightarrow f'\left(\frac{\pi}{6}\right) = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

$f''(x) = -\sin x \Rightarrow f''\left(\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

$f'''(x) = -\cos x \Rightarrow f'''\left(\frac{\pi}{6}\right) = -\cos(\frac{\pi}{6}) = -\frac{\sqrt{3}}{2}$

Thus the third Taylor polynomial is

$$p_3(x) = \sum_{i=0}^{3} \frac{f^i(\frac{\pi}{6})}{i!}.(x-\frac{\pi}{6})^i = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right) + \frac{f''\left(\frac{\pi}{6}\right)}{2!}(x-\frac{\pi}{6})^2 + \frac{f'''\left(\frac{\pi}{6}\right)}{3!}(x-\frac{\pi}{6})^3$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right) - \frac{1}{2}\frac{(x-\frac{\pi}{6})^2}{2!} - \frac{\sqrt{3}}{2}\frac{(x-\frac{\pi}{6})^3}{3!}$$

Hence the desired approximation is

$$\sin x \cong p_3(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x-\frac{\pi}{6}\right) - \frac{1}{4}(x-\frac{\pi}{6})^2 - \frac{\sqrt{3}}{12}(x-\frac{\pi}{6})^3$$

b. $f(x) = e^x$, $a = 2$ and $n = 3$

Solution: We have $f(x) = e^x$ and by definition the Taylor series at $a = 2$ is $f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} . (x - a)^n$.

Step1: Compute all the derivatives $f^n(a)$, $n = 0 , 1 , 2 , 3$

$\Longrightarrow f'(x) = f''(x) = \cdots = f^n(x) = e^x$ for all x .

$\Longrightarrow f^n(a) = f^n(2) = e^2$ for all n.

Step2: Write down the Taylor polynomial of f(x) at $a = 2$

$\Longrightarrow f(x) = \sum_{n=0}^{3} \frac{f^n(a)}{n!} . (x - a)^n$

$= f(a) + f'(a). (x - a) + \frac{f''(a)}{2!} . (x - a)^2 + \frac{f'''(a)}{3!} . (x - a)^3$

$= f(2) + f'(2). (x - 2) + \frac{f''(2)}{2!} . (x - 2)^2 + \frac{f'''(2)}{3!} . (x - 2)^3$

Thus the third Taylor polynomial is

$p_3(x) = e^2 + e^2(x - 2) + e^2 \frac{(x-2)^2}{2!} + e^2 \frac{(x-2)^3}{3!}$

$\Longrightarrow p_3(x) = e^2(1 + (x - 2) + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{6})$

$\Longrightarrow e^x \cong p_3(x) = e^2(1 + (x - 2) + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{6})$

c. $f(x) = \sqrt{3 + x^2}$; $a = 1$ and $n = 2$

Solution: $f(x) = \sqrt{3 + x^2} \Longrightarrow f(1) = 2$

$f'(x) = \frac{x}{\sqrt{3 + x^2}} \Longrightarrow f'(1) = \frac{1}{2}$

$f''(x) = \frac{3}{(3 + x^2).\sqrt{3 + x^2}} \Longrightarrow f''(1) = \frac{3}{8}$

Thus $p_2(x) = 2 + \frac{1}{2}. (x - 1) + \frac{3}{8} . \frac{(x-1)^2}{2!}$

$= 2 + \frac{1}{2}. (x - 1) + \frac{3}{16} . (x - 1)^2$

$\Longrightarrow \sqrt{3 + x^2} \cong p_2(x) = 2 + \frac{1}{2}. (x - 1) + \frac{3}{16} . (x - 1)^2$

d. Express the polynomial $f(x) = 2. x^3 - 9 . x^2 + 11 . x - 1$ as a polynomial in $(x - 2)$.

Solution: we have $a = 2$ and $f(x) = 2. x^3 - 9 . x^2 + 11 . x - 1$

$\Longrightarrow f(2) = 1$ and $f'(x) = 6. x^2 - 18 . x + 11$

$\Longrightarrow f'(2) = -1$

$f''(x) = 12.x - 18$

$\Rightarrow f''(2) = 6$

$f'''(x) = 12$

$\Rightarrow f'''(2) = 12$

In general $f^n(x) = 0$ for all $n \geq 4$ $\Rightarrow f^n(2) = 0$ for all $n \geq 4$

Therefore $f(x) = 1 - (x-2) + \frac{6}{2!}.(x-2)^2 + \frac{12}{3!}(x-2)^3$

$= 1 - (x-2) + 3.(x-2)^2 + 2(x-2)^3$.

Example 15:

1. Find the Taylor series of f about a

   a. $f(x) = 4x^2 - 2x + 1$, $a = 0, 3$

Solution: $f(x) = 4x^2 - 2x + 1 \Rightarrow f(0) = 1$

$f'(x) = 8x - 2 \Rightarrow f'(0) = -2$

$f''(x) = 8 \Rightarrow f''(0) = 8$

In general $f^n(x) = 0$ for all $n \geq 3$ $\Rightarrow f^n(0) = 0$ for all $n \geq 3$

$f(x) = 1 - 2x + \frac{8x^2}{2!} = 1 - 2x + 4x^2$

Similarly by the same method we obtain $f(3) = 31$, $f'(3) = 22$, $f''(3) = 8$

$\Rightarrow f(x) = 31 + 22(x-3) + \frac{8(x-3)^2}{2!} = 31 + 22(x-3) + 4(x-3)^2$

   b. $f(x) = \frac{1}{x}$, $a = -1$

Solution: $f(x) = \frac{1}{x}$ $\Rightarrow f(-1) = -1$

$f'(x) = \frac{-1}{x^2}$ $\Rightarrow f'(-1) = -1$

$f''(x) = \frac{2}{x^3}$ $\Rightarrow f''(-1) = -2$

$f'''(x) = \frac{-6}{x^4}$ $\Rightarrow f'''(-1) = -6$

$f^4(x) = \frac{24}{x^5}$ $\Rightarrow f^4(-1) = -24$

$f^5(x) = \frac{-120}{x^6}$ $\Rightarrow f^5(-1) = -120$

In general $f^n(-1) = -n!$

Thus,

$f(x) = -1 - (x+1) - \frac{2(x+1)^2}{2!} - \frac{6(x+1)^3}{3!} - \frac{24(x+1)^4}{4!} + \cdots$

$= -1 - (x+1) - (x+1)^2 - (x+1)^3 - (x+1)^4 + \cdots$

c. $f(x) = \sin 2x \quad , \quad a = 0$

Solution: $f(x) = \sin 2x \quad \Rightarrow f(0) = \sin 0 = 0$

$f'(x) = 2 \cos 2x \quad \Rightarrow f'(0) = 2 . \cos 0 = 2$

$f''(x) = -4 \sin 2x \quad \Rightarrow f''(0) = -4 . \sin 0 = 0$

$f^3(x) = -8 \cos 2x \quad \Rightarrow f^3(0) = -8 . \cos 0 = -8$

$f^4(x) = 16 \sin 2x \quad \Rightarrow f^4(0) = 16 . \sin 0 = 0$

$f^5(x) = 32 \cos 2x \quad \Rightarrow f^5(0) = 32. \cos 0 = 32$

$f^6(x) = -64 \sin 2x \quad \Rightarrow f^6(0) = -64 . \sin 0 = 0$

$f^7(x) = -128 \cos 2x \quad \Rightarrow f^7(0) = -128 . \cos 0 = -128$

Thus,

$$f(x) = 2x - \frac{8x^3}{3!} + 32\frac{x^5}{5!} - 128\frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty}(-1)^n . \frac{(2x)^{2n+1}}{(2n+1)!}$$

d. $f(x) = \sqrt{x} \quad , \quad a = 1$

Solution: $f(x) = \sqrt{x} = x^{1/2} \quad \Rightarrow f(1) = 1$

$f'(x) = \frac{1}{2\sqrt{x}} \quad \Rightarrow f'(1) = \frac{1}{2}$

$f''(x) = \frac{-1}{4} . x^{-3/2} \quad \Rightarrow f''(1) = \frac{-1}{4}$

$f^3(x) = \frac{3}{8} . x^{-5/2} \quad \Rightarrow f^3(1) = \frac{3}{8}$

$f^4(x) = \frac{-15}{16} . x^{-7/2} \quad \Rightarrow f^4(1) = \frac{-15}{16}$

$f^5(x) = \frac{105}{32} . x^{-9/2} \quad \Rightarrow f^5(1) = \frac{105}{32}$

Thus,

$$f(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{4}.\frac{(x-1)^2}{2!} + \frac{3}{8}.\frac{(x-1)^3}{3!} - \frac{15}{16}.\frac{(x-1)^4}{4!} + \frac{105}{32}.\frac{(x-1)^5}{5!} + \ldots$$

2. If $f(x) = \sin(x^3)$, then find $f^{15}(0)$?

Solution: so far we have the Maclaurin series for

$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

Now $\sin(x^3) = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \cdots$

$\frac{f^{15}(0).x^{15}}{15!} = \frac{x^{15}}{5!}$ from which we obtain $f^{15}(0) = \frac{15!}{5!}$.

3. Find the sixth derivative of $f(x) = \frac{1}{1+x^2}$ at $x = 0$?

Solution: If we try to differentiate directly, we will be hopelessly bogged down at about the third derivative; consequently we need another method by which we can get the sixth derivative of the function and for this we have the Maclaurin series.

Since the Maclaurin series for $\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ ...$

Then we have $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \ ...$

Now $\frac{f^6(0).x^6}{6!} = -x^6$

$f^6(0) = -6! = -720$

4. If $f^n(0) = (n+1)!$ for $n = 1, 2, 3, \ldots$, then find the Maclaurin series for f.

Solution: $f(x) = \sum_{n=0}^{\infty} c_n \,.\, (x-a)^n$ but $a = 0$

$\Rightarrow f(x) = \sum_{n=0}^{\infty} c_n \,.\, x^n$ where $c_n = \frac{f^n(0)}{n!} = \frac{(n+1)!}{n!} = n+1$

$\Rightarrow f(x) = \sum_{n=0}^{\infty} (n+1) \,.\, x^n$

Exercise 3.6

1. *Find the Taylor polynomial up to degree 5 for*
   i. $f(x) = \cos x$ *center at* $a = 0$
   ii. $f(x) = \frac{1}{x}$ *center at* $a = 1$
2. *Let* $f(x) = x^6 - 3x^4 + 2x - 1$
   a. *Find the fifth Taylor polynomial of* f *about 0 ?*
   b. *Find the fourth Taylor polynomial of* f *about -1 ?*
   c. *Find the Taylor series of* f *about -1 ?*

Unit Summary:

1. A series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots \quad \text{or} \quad \sum_{n=0}^{\infty} a_nx^n$$

is called a power series in x or a power series.

- A more generalized form of a power series is of $(x - a)$, that is, an infinite series of the form

  $$a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n + \cdots \quad \text{or} \quad \sum_{n=0}^{\infty} a_n(x - a)^n$$

  is called a power series in $(x - a)$.

  If $a = 0$, this general power series becomes a power series in x.

2. A power series is said to converge at x_0 if the series of real numbers $\sum_{n=0}^{\infty} a_nx^n$ converges at x_0 ; or,

A power series is said to be convergent in a set D of real numbers if it is convergent for every real number x in D.

3. Let $\sum_{n=0}^{\infty} a_n.x^n$ be a power series, then exactly one of the following conditions hold

i. The power series $\sum_{n=0}^{\infty} a_n.x^n$ converges only at $x = 0$. Example: $\sum_{n=0}^{\infty} n!.x^n$

ii. The power series $\sum_{n=0}^{\infty} a_n.x^n$ converges for all x . Example: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

iii. There exists a positive real number R such that the power series $\sum_{n=0}^{\infty} a_n.x^n$

- Converges for all x with $|x| < R$, that is, $-R < x < R$.
- Diverges for all x with $|x| > R$.

4. The algebraic operation on power series are determined

a. If $\sum_{n=0}^{\infty} c_n . s^n$ converges, then $\sum_{n=0}^{\infty} c_n . x^n$

- Converges absolutely for $|x| < |s|$
- Diverges when $|x| > |s|$

b. If $\sum_{n=0}^{\infty} c_n . s^n$ diverges , then $\sum_{n=0}^{\infty} c_n . x^n$

- Converges absolutely for $|x| \le |s|$
- Diverges for $|x| > |s|$

5. **Differentiation Theorem for Power Series**: If the power series $\sum_{n=0}^{\infty} c_n\,(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is differentiable, and

$$f'(x) = (c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots)' = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots$$

Or,

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} c_n\,(x-a)^n\right) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} .$$

Similarly, the derivative for a power series $f(x) = \sum_{n=0}^{\infty} a_n\,x^n$ with a radius of convergence $R > 0$ is given by:

$$f'(x) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n\,x^n\right) = \sum_{n=0}^{\infty} \frac{d}{dx}(a_n\,x^n) = \sum_{n=1}^{\infty} na_n\,x^{n-1}$$

$$= a_1 + 2a_2x + 3a_3x^3 + \cdots$$

6. **Integration Theorem for a Power Series**: If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = \sum_{n=0}^{\infty} c_n\,(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is integrable, and

$$\int f(x)dx = \int \sum_{n=0}^{\infty} c_n\,(x-a)^n\,dx = \sum_{n=0}^{\infty} \int c_n\,(x-a)^n dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}(x-a)^{n+1} + k$$

where k is a constant.

Similarly, if a power series $f(x) = \sum_{n=0}^{\infty} c_n\,x^n$ has a radius of convergence $R > 0$, then it is integrable, and it is given by:

$$\int f(x)dx = \int \sum_{n=0}^{\infty} c_n\,x^n\,dx = \sum_{n=0}^{\infty} \int c_n\,x^n dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1}x^{n+1} + k$$

7. If f has a power series representation at a , that is, if $f(x) = \sum_{n=0}^{\infty} c_n\,(x-a)^n$, $|x-a| < R$, then its coefficients are given by $c_n = \frac{f^n(a)}{n!}$.

8. **<u>Taylor series</u>:** If f has a power series representation at a , then any expression of the form: $f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} . (x-a)^n = f(a) + \frac{f'(a)}{1!} . (x-a) + \frac{f''(a)}{2!} . (x-a)^2 + \frac{f'''(a)}{3!} . (x-a)^3 + \cdots$ is called the Taylor series of the function f at a .

If $a = 0$, then $f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} . x^n = f(0) + \frac{f'(0)}{1!} . x + \frac{f''(0)}{2!} . x^2 + \frac{f'''(0)}{3!} . x^3 + \cdots$ is called the Maclaurin series.

9. Some Useful Maclaurin Series

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$
- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
- $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$
- $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$

10. <u>Taylor polynomial</u>: Taylor polynomials are applicable in approximating functions because polynomials are the simplest of functions.

If f(x) is equal to the sum of its Taylor series at a, that is, $f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} . (x - a)^n$, then

- The n^{th} - partial sum of the Taylor series, denoted by $P_n(x)$, is called the n^{th} degree Taylor polynomial of f at a and is given by

  $$p_n(x) = \sum_{i=0}^{n} \frac{f^i(a)}{i!} . (x - a)^i = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^n(a)}{n!} . (x - a)^n.$$

- $p_n(x)$ can be used as an approximation to f i.e. $f(x) \cong p_n(x)$.

Miscellaneous Exercises

1. *For each of the following series, find the radius of convergence(R) and the interval of Convergence*

a. $\sum_{n=0}^{\infty} \frac{(-1)^n . x^n}{4^n . \ln n}$ b. $\sum_{n=0}^{\infty} \frac{(-1)^n . x^{2n}}{(2n)!}$ c. $\sum_{n=1}^{\infty} \frac{(-1)^n . (x+2)^n}{n . 2^n}$

d. $\sum_{n=0}^{\infty} n! \ (2x-1)^n$ e. $\sum_{n=1}^{\infty} \frac{x^n}{2.4.6....(2n)}$ f. $\sum_{n=2}^{\infty} (-1)^n . \frac{(\ln n)^2 . x^n}{n^2}$

g. $\sum_{n=0}^{\infty} \frac{3^n}{5^{2n}} . x^{3n}$ h. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} . x^{2n}$ i. $\sum_{n=0}^{\infty} \frac{n^{2n}}{(2n)!} . x^n$

2. *Find* $f'(x)$, $f''(x)$ *and* $\int f(x)dx$ *for the following power series*

a. $f(x) = \sum_{n=1}^{\infty} n^{-3} . x^n$ b. $f(x) = \sum_{n=1}^{\infty} \frac{n!}{n^n} . x^n$ c. $f(x) = \sum_{n=2}^{\infty} \frac{x^n}{\ln n}$

d. $f(x) = \sum_{n=1}^{\infty} (-1)^n . \frac{n+1}{3^n} . x^n$ e. $f(x) = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n$ f. $\sum_{n=1}^{\infty} \frac{(x-1)^{n+1}}{n+1}$

3. *If* $\sum_{n=0}^{\infty} c_n 4^n$ *is convergent, then does it follow that the following series are convergent?*

a. $\sum_{n=0}^{\infty} c_n . (-2)^n$ b. $\sum_{n=0}^{\infty} c_n . (-4)^n$

4. *Suppose that* $\sum_{n=0}^{\infty} c_n . x^n$ *converges when* $x = -4$ *and diverges when* $x = 6$. *What can be said about the convergence or divergence of the following series ?*

a. $\sum_{n=0}^{\infty} c_n$ b. $\sum_{n=0}^{\infty} c_n . 8^n$ c. $\sum_{n=0}^{\infty} c_n . (-3)^n$ d. $\sum_{n=0}^{\infty} (-1)^n . c_n . 9^n$

5. *If* k *is a positive integer, then find the radius of convergence of the series*

$\sum_{n=0}^{\infty} \frac{(n!)^k . x^n}{(kn)!}$.

6. *Suppose that the series* $\sum_{n=0}^{\infty} c_n . x^n$ *has radius of convergence 2 and the series*

$\sum_{n=0}^{\infty} d_n \, . x^n$ has radius of convergence 3, what is the radius of convergence of the series $\sum_{n=0}^{\infty} (c_n + d_n) \, . x^n$?

7. *Suppose that the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n \, . x^n$ is R, what is the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n \, . x^{2n}$?*

8. *If the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n \, . x^n$ is 10, what is the radius of convergence of the series $\sum_{n=1}^{\infty} n \, . c_n \, . x^{n-1}$? $\sum_{n=1}^{\infty} \frac{c_n}{n+1} \, . x^{n+1}$? Why ?*

9. *If $f(x) = \sum_{n=0}^{\infty} c_n \, . (x-2)^n$ for all n, then what is the formula for*

e. c_5 *b.* c_9 *c.* c_n

10. *If $f^n(0) = (n+1)!$ for all $n = 1,2,3, \ldots$, then find the Maclaurin series for f and its radius of convergence.*

11. *Find the Taylor series for f centered at 4 if $f^n(4) = \frac{(-1)^n \, . \, n!}{3^n \, . \, (n+1)}$. What is the radius of convergence of the Taylor series ?*

12. *Find the Taylor series for $f(x)$ centered at the given values of a*

- **a.** $f(x) = 5 + 2x + 3x^2 + x^3 + x^4$; *a=2*
- **b.** *b.* $f(x) = \cos x$; *a=π*
- **c.** $f(x) = \ln x$; *a=2*
- **d.** *c.* $f(x) = \sin x$; $a = \pi/2$
- **e.** *e.* $f(x) = x^{-2}$; *a=1*

13. *Find the Taylor polynomial up to degree 5 for*

- **i.** $f(x) = \cos x$ *center at* $a = 0$.
- **ii.** $f(x) = \frac{1}{x}$ *center at* $a = 1$.

14. *Let* $f(x) = x^6 - 3x^4 + 2x - 1$

- ***a.*** *Find the fifth Taylor polynomial of f about 0.*

b. *Find the fourth Taylor polynomial of f about -1.*

c. *Find the Taylor series of f about -1.*

15. *Find the respective derivative of the following functions*

a. $f^{(6)}(0)$ *where* $f(x) = \frac{1}{1+x^2}$?

b. $f^{(4)}(0)$ *where* $f(x) = \frac{1}{1-2x^3}$?

c. $f^{(6)}(0)$ *where* $f(x) = x.e^x$?

d. $f^{(15)}(0)$ *where* $f(x) = sin(x^3)$?

e. $f^{(5)}(0)$ *where* $f(x) = \frac{x}{1+x^2}$?

f. $f^{(8)}(0)$ *where* $f(x) = cos(x^2)$?

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Chapter Four

Differential calculus of function of several variables

Introduction

In this chapter we consider the integral of a function of two variables $f(x, y)$ over a region in the plane and the integral of a function of three variables $f(x, y, z)$ over a region in space. These integrals are called multiple integrals and are defined as the limit of approximating Riemann sums, much like the single – variable integrals. We can use multiple integrals to calculate quantities that vary over two or three dimensions, such as the total mass or the angular momentum of an object of varying density and the volumes of solids with general curved boundaries like volumes of hyperspheres. In addition, we can use double and triple integrals to compute probabilities, average temperature and so on.

We will see that polar coordinates are useful in computing double integrals over some types of regions. In a similar way, we will introduce two coordinate systems in three – dimensional space which are cylindrical coordinates and spherical coordinates that greatly simplify the computation of triple integrals over certain commonly occurring solid regions.

Unit Objectives:

On the completion of this unit, students should be able to:

- Understand functions of several variables
- Find domain and range of function of several variables
- Sketch graphs of functions of several variables
- Find level curves of functions of several variables
- Find limits of functions of several variables
- Understand the idea of continuity in case of several variables
- Find partial derivatives
- Apply partial derivatives
- Find directional derivatives and gradients
- Find tangent planes to functions of several variables

- Understand tangent plane approximation
- Find relative extrema to functions of several variables
- Understand the idea of Lagrange multiplier

4.1. Functions of Several variables

Overview

In this section we will see the definition of functions of two and three variables, domains and ranges of functions of two and three variables and graphs of those functions, level curves and level surfaces of functions.

Section objective:

After the completion of this section, successful students be able to:

- Define functions of two and three variables
- Find domain and range of functions of two and three variables
- Find level curves and level surfaces of given functions

Definition 4.1: A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is the set of values that f takes on, that is, $\{f(x, y)|(x, y)\epsilon D\}$. Here, the variables x and y are **independent variables** and f is the **dependent variable**.

Example 1: For each of the following functions, evaluate $f(3,2)$ and find the domain.

a. $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$ b. $f(x, y) = x\ln(y^2 - x)$

Solution:

a. $f(3,2) = \frac{\sqrt{3+2+1}}{3-1} = \frac{\sqrt{6}}{2}$ and

The expression for f makes sense if the denominator is not 0 and the quantity under the square root sign is nonnegative. So the domain of f is

$$D = \{(x, y)|x + y + 1 \geq 0, x \neq 1\}$$

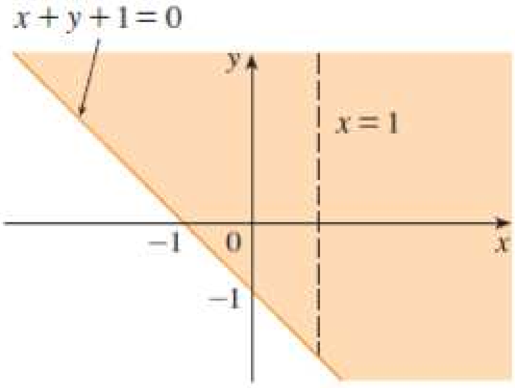

Figure 1 Domain of $f(x, y = \frac{\sqrt{x+y+1}}{x-1})$

b. $f(3,2) = 3\ln((2)^2 - 3) = 3\ln 1 = 0$ and

Since $\ln(y^2 - x)$ is defined only when $y^2 - x > 0$, that is, $x < y^2$, the domain of f is

$$D = \{(x, y)|x < y^2\}$$

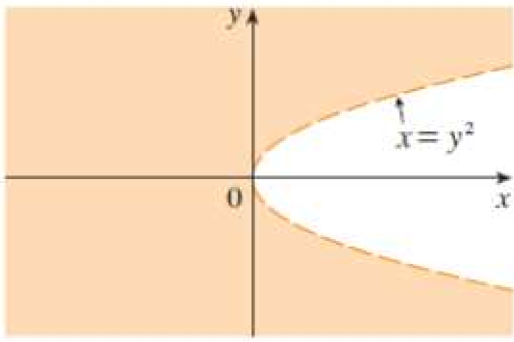

Figure 2 Domain of $f(x, y = x\ln(y^2 - x)$

Example 2: Find the domain and range of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution: The domain of g is

$$D = \{(x, y)|9 - x^2 - y^2 \geq 0\} = \{(x, y)|x^2 + y^2 \leq 9\}$$

which is the disk with center $(0,0)$ and radius 3 and the range of g is

$$\{z|z = \sqrt{9 - x^2 - y^2}, (x, y)\epsilon D\}$$

Since z is a positive square root, $z \geq 0$ and $9 - x^2 - y^2 \leq 9$

$$\Rightarrow \sqrt{9 - x^2 - y^2} \leq 3$$

Therefore, the range of g is

$$\{z \mid 0 \leq z \leq 3\} = [0,3].$$

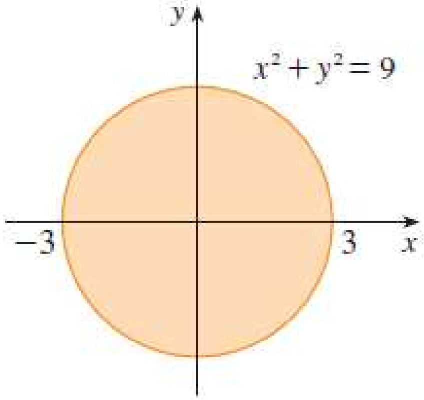

Figure 3 Domain of $g(x, y = \sqrt{9 - x^2 - y^2}$

4.1.1. Graphs

> **Definition 4.2:** If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Example 3: Sketch the graph of the function $f(x, y) = 6 - 3x - 2y$.

Solution: The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$, which represents a plane. To graph the plane we first find the intercepts.

Putting $y = z = 0$ in the equation, we get $x = 2$ as the x-intercept. Similarly, the $y -$ intercept is 3 and the $z -$ intercept is 6.

Therefore, the graph of $f(x, y) = 6 - 3x - 2y$ is given below.

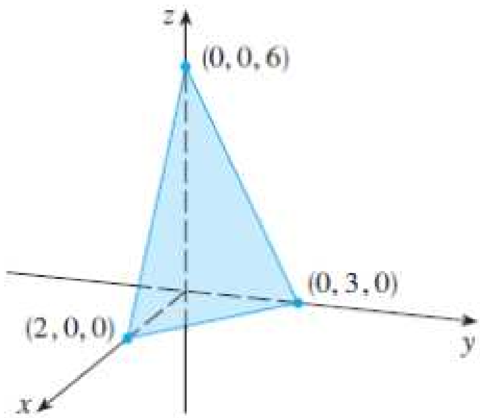

Figure 4

Example 4: Sketch the graph of the function $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution: The graph of f has the equation $z = \sqrt{9 - x^2 - y^2}$. We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$, which represents a sphere with center the origin and radius 3. But, since $z \geq 0$, the graph of g is just the top half of the sphere as in figure 5 below.

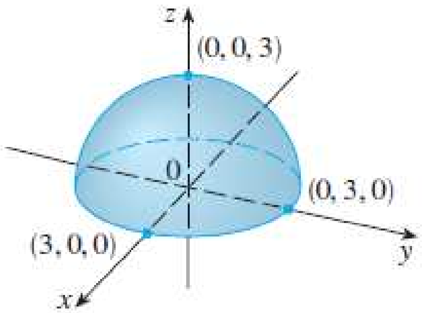

Figure 5

4.1.2. Level curves

Definition 4.3:

1. The **level curves (contour curves)** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant.
2. The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = k$ is called a level curve of f. Here k is a constant.

Example 5: Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6$ and 12.

Solution: The level curves are

$$6 - 3x - 2y = k \text{ or } 3x + 2y + (k - 6) = 0$$

This is a family of lines with slope $-\frac{3}{2}$. The four particular level curves with $k = -6, 0, 6$ and 12 are:

$$3x + 2y - 12 = 0 \text{ for } k = -6$$

$$3x + 2y - 6 = 0 \text{ for } k = 0$$

$3x + 2y = 0$ for $k = 6$

$3x + 2y + 6 = 0$ for $k = 12$.

They are sketched in Figure 6 and they are equally spaced parallel lines because the graph of f is a plane.

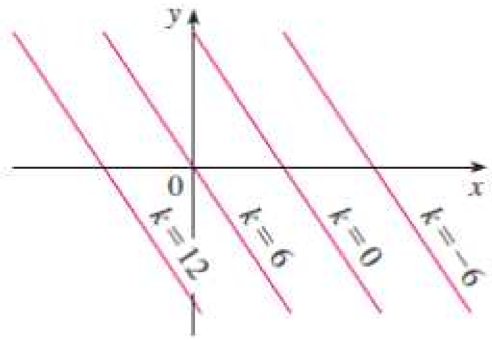

Figure 6 Contour curves of $f(x, y) = 6 - 3x - 2y$

Example 6: Sketch the level curves of the function

$$g(x, y) = \sqrt{9 - x^2 - y^2} \text{ for } k = 0,1,2,3.$$

Solution: The level curves are

$$\sqrt{9 - x^2 - y^2} = k \text{ or } x^2 + y^2 + (k^2 - 9) = 0$$

This is a family of concentric circles with center $(0,0)$ and radius $\sqrt{9 - k^2}$. The four particular level curves with $k = 0,1,2$ and 3 are:

$x^2 + y^2 - 9 = 0$ for $k = 0$

$x^2 + y^2 - 8 = 0$ for $k = 1$

$x^2 + y^2 - 5 = 0$ for $k = 2$

$x^2 + y^2 = 0$ for $k = 3$.

Therefore, the graph of these level curves is given in figure 7 below.

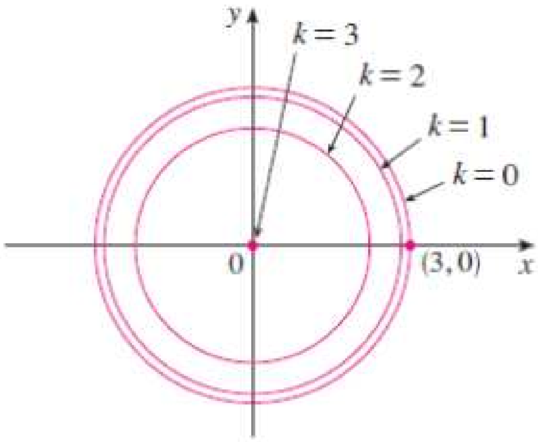

Figure 7 Level curves of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Functions of three variables

Definition 4.4: A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \epsilon \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.

Example 7: Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

Solution: The expression for $f(x, y, z)$ is defined as long as $z - y > 0$. So, the domain of f is

$$D = \{(x, y, z) | z > y\}$$

Remark: It's very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space. However, we do gain some insight into by examining its **level surfaces.**

Definition 4.5: The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = k$ is called a **level surface** of f.

Example 8: Find the level surfaces of the function

$f(x, y, z) = x^2 + y^2 + z^2$ for $k = 1,4$ and 9.

Solution: The level surfaces are

$$x^2 + y^2 + z^2 = k, \text{ where } k \geq 0.$$

This is a family of concentric spheres with radius $\sqrt{k}$. The three particular level surfaces with $k = 1,4$ and 9 are

$$x^2 + y^2 + z^2 - 1 = 0 \text{ for } k = 1$$

$$x^2 + y^2 + z^2 - 4 = 0 \text{ for } k = 4$$

$$x^2 + y^2 + z^2 - 9 = 0 \text{ for } k = 9.$$

Therefore, the graph of these level surfaces is given in figure 8 below.

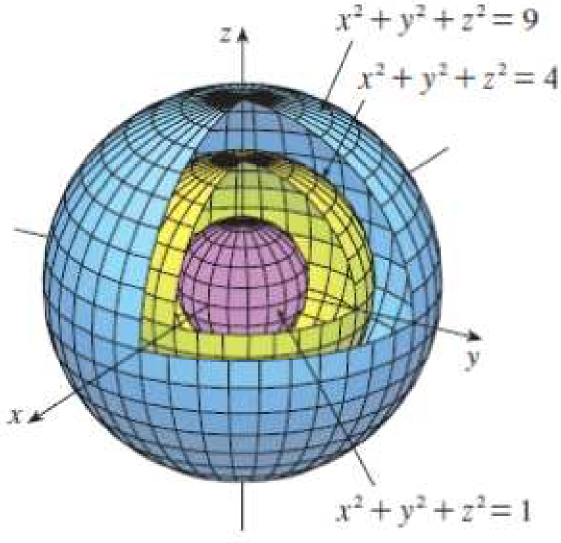

Figure 8

Note 4.1: In general, a **function of *n* variables** is a rule that assigns a number $z = f(x_1, x_2, \dots, x_n)$ to an $n-$ tuple $(x_1, x_2, \dots, x_n)$ of real numbers.

Exercise 4.1

1. *Find the domain and range of the following functions*
   a. $f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$
   b. $f(x, y) = \ln(9 - x^2 - 9y^2)$
   c. $f(x, y) = \sqrt{x + y}$
2. *Sketch the graph of the following functions*

a. $f(x,y) = 10 - 4x - 5y$

b. $f(x,y) = \sqrt{16 - x^2 - 16y^2}$

3. *Sketch the level curve of* $f(x,y) = \sqrt{36 - 9x^2 - 4y^2}$

4. *Find the level surfaces of the following functions*

a. $f(x,y,z) = x + 3y + 5z$

b. $f(x,y,z) = x^2 + 3y^2 + 5z^2$

c. $f(x,y,z) = x^2 - y^2 + z^2$

4.2. Limit and Continuity

Overview

In this section we study the definition of limit and continuity in case of functions of several variables and will do different examples

Section objective:

After the completion of this section, successful students be able to:

- Define limit and continuity
- Find limits of different functions
- Find points of continuity for different functions

4.2.1. Limit

Definition 4.6: Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then, we say that the limit of $f(x,y)$ as (x,y) approaches (a,b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

If for every $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

If $(x,y)\epsilon D$and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$

Example 9: Find $\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2}$ if it exists.

Solution: We first observe that along the line $x = 0$, the function always has value 0 when $y \neq 0$. Likewise, along the line $y = 0$ the function has value 0 provided $x \neq 0$. So if the limit does exist as (x, y) approaches (0, 0), the value of the limit must be 0. To see if this is true, we apply the definition of limit.

Let $\varepsilon > 0$ be given, but arbitrary. We want to find a $\delta > 0$ such that

If $0 < \sqrt{x^2+y^2} < \delta$, then $\left|\frac{4xy^2}{x^2+y^2} - 0\right| < \varepsilon$

$\Rightarrow 0 < \sqrt{x^2+y^2} < \delta$, then $\frac{4|x|y^2}{x^2+y^2} < \varepsilon$

Since $y^2 \leq x^2 + y^2$ we have that

$$\frac{4|x|y^2}{x^2+y^2} \leq \frac{4|x|y^2}{y^2} = 4|x| = 4\sqrt{x^2} \leq 4\sqrt{x^2+y^2} < \varepsilon \text{ (Since } x^2 \leq x^2 + y^2\text{)}$$

So if we choose $\delta = {}^{\varepsilon}/_{4}$ and let $0 < \sqrt{x^2+y^2} < \delta$, we get

$$\left|\frac{4xy^2}{x^2+y^2} - 0\right| \leq 4\sqrt{x^2+y^2} < 4\delta$$

$$= 4\left({}^{\varepsilon}/_{4}\right) = \varepsilon$$

It follows from the definition that

$$\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2+y^2} = 0$$

Theorem 4.1: Properties of Limits of Functions of Two Variables

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(a,b)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(a,b)} g(x,y) = M$$

1. Sum and Difference rule: $\lim_{(x,y)\to(a,b)}(f(x,y) \pm g(x,y) = L \pm M$

2. Product rule: $\lim_{(x,y)\to(a,b)}(f(x,y).g(x,y)) = L.M$

3.Constant multiple rule: $\lim_{(x,y)\to(a,b)}\big(kf(x,y)\big) = kL, (any\ number\ k)$

4.Quotient rule: $\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, M \neq 0$

5.Power rule: If *r* and *s* are integers with no common factors, and $s \neq 0$, then

$$\text{Lim}_{(x,y)\to(a,b)}\big(f(x,y)\big)^{r/s} = L^{r/s}$$

provided $L^{r/s}$ is a real number. (If *s* is even, we assume that $L > 0$.)

Example 10: Find $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$ if it exists

Solution: Since the denominator $\sqrt{x}-\sqrt{y}$ approaches 0 as $(x,y)\to(0,0)$ we cannot use the Quotient Rule from the above theorem. If we multiply numerator and denominator by $\sqrt{x}+\sqrt{y}$, however, we produce an equivalent fraction whose limit we *can* find:

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{(x^2-xy)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x(x-y)(\sqrt{x}+\sqrt{y})}{x-y}$$

$$= \lim_{(x,y)\to(0,0)} x\big(\sqrt{x}+\sqrt{y}\big)$$

$$= 0(\sqrt{0}+\sqrt{0})$$

$$= 0$$

We can cancel $(x-y)$the factor because the path $y = x$ (along which $x - y = 0$) is *not* in the domain of the function $\frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$. .

Remark 4.1: If $\lim_{(x,y)\to(a,b)} f(x,y) = L_1$ along a path C_1 and $\lim_{(x,y)\to(a,b)} f(x,y) = L_2$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example 11: If $f(x,y) = \frac{xy}{x^2+y^2}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution: If $y = 0$, then $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to0} f(x,0) = \lim_{x\to0} \frac{x.0}{x^2+0^2} = 0$

$f(x,y) \to 0\ as\ (x,y)\to(0,0)$ along the x −axis

If $x = 0$, then $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0.y}{0^2+y^2} = 0$

$$f(x,y) \to 0 \; as \; (x,y) \to (0,0) \text{ along the } y\text{-axis}$$

Although we have obtained identical limits along the axes. But that does not show that the given limit is 0. Let's now approach $(0,0)$ along another line, say $y = x$. For all $x \neq 0$, then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,x) = \lim_{y\to 0} \frac{x.x}{x^2+x^2} = \frac{1}{2}$$

$$f(x,y) \to \frac{1}{2} \; as \; (x,y) \to (0,0) \text{ along } y = x.$$

(See Figure 9.) Since we have obtained different limits along different paths, the given limit does not exist.

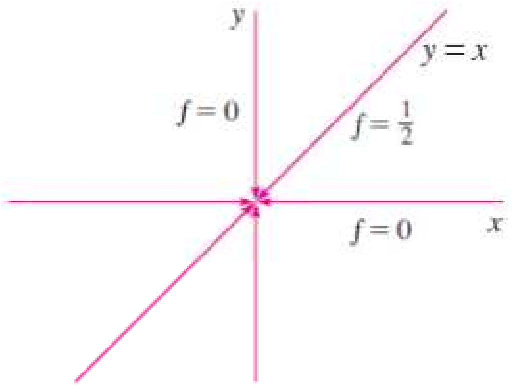

Figure 9

Example 12: If $f(x,y) = \frac{xy^2}{x^2+y^4}$, does $\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

Solution: If $y = 0$, then $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x.0}{x^2+0^4} = 0$

$$f(x,y) \to 0 \; as \; (x,y) \to (0,0) \text{ along the } x\text{-axis}$$

If $x = 0$, then $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0.y^2}{0^2+y^4} = 0$

$$f(x,y) \to 0 \; as \; (x,y) \to (0,0) \text{ along the } y\text{-axis}$$

If $y = mx$,where m is the slop, then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,mx) = \lim_{x\to 0} \frac{x.(mx)^2}{x^2+(mx)^4} = 0$$

$$f(x,y) \to 0 \; as \; (x,y) \to (0,0) \text{ along } y = mx.$$

Although we have obtained identical limits along the given lines. But that does not show that the given limit is 0. Let's now approach $(0,0)$ along the parabola $x = y^2$, then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(y^2, y) = \lim_{y\to 0} \frac{y^2 . y^2}{y^4 + y^4} = \frac{1}{2}$$

$$f(x,y) \to \frac{1}{2} \; as \; (x,y) \to (0,0) \text{ along } x = y^2.$$

Since different paths lead to different limiting values, the given limit does not exist.

4.2.2. Continuity

Definition 4.7: A function $f(x,y)$ is **continuous at the point (a,b)** if

1. f is defined at (a,b),
2. $\lim_{(x,y)\to(a,b)} f(x,y)$ exists,
3. $\lim_{(x,y)\to(0,0)} f(x,y) = f(a,b)$.

✓ A function is **continuous** if it is continuous at every point of its domain.

Example 13: Show that

$$f(x,y) = \begin{cases} \frac{3x^2 y}{x^2 + y^2}, if \; (x,y) \neq (0,0) \\ 0, \qquad if \; (x,y) = (0,0) \end{cases}$$

is continuous on $\mathbb{R}^2$.

Solution: We know f is continuous for $(x,y) \neq (0,0)$, since it is equal to a rational function there. Now, let $y = mx, m \neq 0$, then

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x, mx)$$

$$= \lim_{x\to 0} \frac{3x.(mx)^2}{x^2 + (mx)^2}$$

$$= \lim_{x\to 0} \frac{3mx}{1+m^2}$$

$$= 0$$

This shows that, $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$.

Therefore, f is continuous $(0,0)$, and so it is continuous on $\mathbb{R}^2$.

Example 14: Show that

$$f(x,y)=\begin{cases}\frac{2xy}{x^2+y^2}, if\ (x,y)\neq(0,0)\\ 0, \quad\quad if\ \ (x,y)=(0,0)\end{cases}$$

is continuous at every point except the origin.

Solution: The function f is continuous at any point $(x,y)\neq(0,0)$, because its values are then given by a rational function of x and y.

At (0, 0), the value of f is defined, but f, we claim, has no limit as $(x,y)\to(0,0)$. The reason is that different paths of approach to the origin can lead to different results, as we now see.

Let $y=mx, m\neq 0$, then

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{x\to 0} f(x,mx)$$

$$=\lim_{x\to 0}\frac{2x.mx}{x^2+(mx)^2}$$

$$=\frac{2m}{1+m^2}$$

This shows that the limit changes with m.therefore, the limit of the function f as $(x,y)\to(0,0)$ is not unique and hence the limit does not exist. Thus, the function f is not continuous at the origin.

Remark 4.2:

1. If f is continuous at (a,b) and g is a single-variable function continuous at $f(a,b)$, then the composite function $h=gof$ defined by $h(x,y)=g(f(x,y))$ is continuous at (a,b).
2. The definitions of limit and continuity for functions of two variables and the conclusions about limits and continuity for sums, products, quotients, powers, and composites all extend to functions of three or more variables.

Exercise 4.2

1. *Find the limit of the following functions if it exists*

   a. $lim_{(x,y)\to(1,2)}(5x^3-x^2y^2)$

   b. $lim_{(x,y)\to(2,1)}\frac{4-xy}{x^2+3y^2}$

   i. $\lim_{(x,y)\to(0,0)}\frac{e^y\sin x}{x}$

   j. $\lim_{(x,y)\to(1,0)}\frac{x\sin y}{x^2+1}$

c. $lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right)$

d. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$

e. $\lim_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2}$

f. $\lim_{\substack{(x,y)\to(2,2)\\ x\neq y}} \frac{x+y-4}{\sqrt{x+y}-2}$

g. $\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-y^2}{x-y}$

k. $\lim_{(x,y)\to(\pi/2,0)} \frac{\cos y+1}{y-\sin x}$

l. $\lim_{(x,y,z)\to(1,-1,-1)} \frac{2xy+yz}{x^2+z^2}$

m. $\lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{xy-y-2x+2}{x-1}$

h. $\lim_{\substack{(x,y)\to(2,0)\\ 2x-4\neq y}} \frac{\sqrt{2x-y}-2}{2x-y-4}$

i. $\lim_{(x,y,z)\to(1,3,4)} \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

2. *By considering different paths of approach, show that the following functions have no limits as* $(x,y)\to(0,0)$.

a. $f(x,y) = \frac{x^4}{x^4+y^2}$

b. $f(x,y) = \frac{x^4-y^2}{x^4+y^2}$

c. $f(x,y) = \frac{xy}{|xy|}$

d. $h(x,y) = \frac{x^2}{x^2-y}$

e. $h(x,y) = \frac{x^2+y}{y}$

f. $h(x,y) = \frac{x-y}{x+y}$

3. *Find the point of continuity for the following functions*

a. $f(x,y) = \sin(x+y)$

b. $g(x,y) = \frac{1}{x^2-y}$

c. $f(x,y) = \frac{y}{x^2+1}$

d. $g(x,y,z) = xy\sin\frac{1}{z}$

e. $h(x,y) = \frac{x+y}{2+\cos x}$

f. $h(x,y,z) = e^{x+y}\cos z$

4.3. Partial Derivatives

Overview

In this section we will define the different (order) partial derivatives of functions of several variables and find those partial derivatives for different functions, again we will define total differential and give examples.

Section objective:

After the completion of this section, successful students be able to:

- Define partial derivative and total differential
- Find partial and total derivatives for different functions

Definition 4.8:

1. Partial derivative with respect to x

   The **partial derivative of $f(x, y)$ with respect to x** at the point (a, b) is

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \left.\frac{d}{dx} f(x, b)\right|_{x=a} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

   provided the limit exists.

2. Partial derivative with respect to y

   The **partial derivative of $f(x, y)$ with respect to y** at the point (a, b) is

$$\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = \left.\frac{d}{dy} f(a, y)\right|_{y=b} = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}$$

   provided the limit exists.

Notations for partial derivatives: If the function is $f = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = f_2 = D_2 f = D_y f$$

Remark 4.3: Let $f = f(x, y)$, then

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example 15: Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$

Solution: To find $\frac{\partial f}{\partial x}$ we consider y as a constant and differentiate with respect to x:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1)$$

$$= 2x + 3y + 0 - 0$$

$$= 2x + 3y$$

Then, $\left.\frac{\partial f}{\partial x}\right|_{(4,-5)} = 2(4) + 3(-5) = -7.$

To find $\frac{\partial f}{\partial y}$ we consider x as a constant and differentiate with respect to y:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 + 3xy + y - 1)$$

$$= 0 + 3x + 1 - 0$$

$$= 3x + 1$$

Then, $\left.\frac{\partial f}{\partial y}\right|_{(4,-5)} = 3(4) + 1 = 13.$

Example 16: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if

$$f(x, y) = \frac{2y}{y+\cos x}$$

Solution: We treat f as a quotient. With y held constant, we get

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2y}{y+\cos x}\right) = \frac{(y+\cos x)\frac{\partial}{\partial x}(2y) - 2y\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)(0) - 2y(-\sin x)}{(y+\cos x)^2}$$

$$= \frac{2y\sin x}{(y+\cos x)^2}.$$

With x held constant, we get

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right) = \frac{(y+\cos x)\frac{\partial}{\partial y}(2y) - 2y\frac{\partial}{\partial y}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)(2) - 2y(1)}{(y+\cos x)^2}$$

$$= \frac{2\cos x}{(y+\cos x)^2}.$$

Functions of more than two variables

Partial derivatives can also be defined for functions of three or more variables. For example, if f is a function of three variables x, y and z, then its partial derivative with respect to $x, y \; and \; z$ is defined as

$$f_x(x, y, z) = \frac{\partial}{\partial x} f(x, y, z) = \lim_{h\to 0} \frac{f(x+h,y,z) - f(x,y,z)}{h}$$

$$f_y(x, y, z) = \frac{\partial}{\partial y} f(x, y, z) = \lim_{h\to 0} \frac{f(x,y+h,z) - f(x,y,z)}{h}$$

$$f_z(x, y, z) = \frac{\partial}{\partial z} f(x, y, z) = \lim_{h\to 0} \frac{f(x,y,z+h) - f(x,y,z)}{h}$$

In general, if f is a function of n variables, $f = f(x_1, x_2, \dots, x_n)$, then its partial derivative with respect to the i^{th} variable x_i is

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1,x_2,\dots,x_i+h,x_{i+1},\dots,x_n) - f(x_1,x_2,\dots,x_i,\dots,x_n)}{h}$$

Example 17: find f_x, f_y and f_z, if $f(x,y,z) = e^{xy}\ln z$.

Solution: Holding y and z constant and differentiating with respect to x, we have

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^{xy}\ln z) = \ln z\frac{\partial}{\partial x}(e^{xy}) + e^{xy}\frac{\partial}{\partial x}(\ln z)$$
$$= \ln z \,.\, ye^{xy} + e^{xy}(0)$$
$$= ye^{xy}\ln z.$$

Holding x and z constant and differentiating with respect to y, we have

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(e^{xy}\ln z) = \ln z\frac{\partial}{\partial y}(e^{xy}) + e^{xy}\frac{\partial}{\partial y}(\ln z)$$
$$= \ln z \,.\, xe^{xy} + e^{xy}(0)$$
$$= xe^{xy}\ln z.$$

and holding x and y constant and differentiating with respect to z, we have

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(e^{xy}\ln z) = \ln z\frac{\partial}{\partial z}(e^{xy}) + e^{xy}\frac{\partial}{\partial z}(\ln z)$$
$$= \ln z\,(0) + e^{xy}\left(\frac{1}{z}\right)$$
$$= \frac{e^{xy}}{z}.$$

4.3.1. Higher order partial derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$ and $(f_y)_y$, which are called the **second order partial derivatives** of f. If $f = f(x,y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$
$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\partial x}$$
$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\partial y}$$
$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Remark 4.4: Thus, the notation f_{xy}(or $\frac{\partial^2 f}{\partial y\partial x}$) means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

Example 18: Find the second order partial derivatives of

$$f(x,y) = x^3 + x^2y^3 - 2y^2$$

Solution: First we find the first order partial derivatives

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3 + x^2y^3 - 2y^2)$$
$$= 3x^2 + 2xy^3 - 0$$
$$= 3x^2 + 2xy^3$$
$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3 + x^2y^3 - 2y^2)$$
$$= 0 + 3x^2y^2 - 4y$$
$$= 3x^2y^2 - 4y$$

Therefore, the second order partial derivatives are:

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 + 2xy^3)$$
$$= 6x + 2y^3$$
$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(3x^2 + 2xy^3)$$
$$= 6xy^2$$
$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2y^2 - 4y)$$
$$= 6xy^2$$
$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^2y^2 - 4y)$$
$$= 6x^2y - 4$$

Theorem 4.2: Clairaut's Theorem (The Mixed Derivative Theorem): If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Remark 4.5:

1. Partial derivatives of order 3 or higher can also be defined similarly. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right) = \frac{\partial^3 f}{\partial^2 y \partial x}$$

2. Using Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{xyy}$ if these functions are continuous.

Example 19: Find f_{xxyz} if $f(x, y, z) = \sin(3x + yz)$.

Solution:

$$f_x = \frac{\partial}{\partial x}(\sin(3x + yz))$$
$$= 3\cos(3x + yz)$$
$$f_{xx} = \frac{\partial}{\partial x}(3\cos(3x + yz))$$
$$= -9\sin(3x + yz)$$
$$f_{xxy} = \frac{\partial}{\partial y}(-9\sin(3x + yz))$$
$$= -9z\cos(3x + yz)$$
$$f_{xxyz} = \frac{\partial}{\partial z}(-9z\cos(3x + yz))$$
$$= -9\cos(3x + yz) + 9yz\sin(3x + yz)$$

4.3.2. Total Differential

This is a very short section and is here simply to acknowledge that just like we had differentials for functions of one variable we also have them for functions of more than one variable.

Definition 4.9: Given the function $z = f(x, y)$ the differential dz or df is given by:

$$dz = f_x dx + f_y dy \qquad \text{or} \qquad df = f_x dx + f_y dy$$

Similarly, this is extended to functions of three or more variables. For instance, given the function $w = g(x, y, z)$ the differential is given by:

$$dw = g_x dx + g_y dy + g_z dz$$

Example 20: Compute the differentials for each of the following functions.

a. $z = e^{x^2+y^2}\tan(2x)$

b. $u = \frac{t^3 r^6}{s^2}$

Solution:

a. $z = e^{x^2+y^2}\tan(2x)$

The differential of the function is given by:

$$dz = f_x dx + f_y dy$$

But, $f_x = \frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x}\left(e^{x^2+y^2}\tan(2x)\right)$

$$= 2xe^{x^2+y^2}\tan(2x) + 2e^{x^2+y^2}\sec^2(2x)$$

and $f_y = \frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\left(e^{x^2+y^2}\tan(2x)\right)$

$$= 2ye^{x^2+y^2}\tan(2x)$$

Therefore,

$$dz = \left(2xe^{x^2+y^2}\tan(2x) + 2e^{x^2+y^2}\sec^2(2x)\right)dx + \left(2ye^{x^2+y^2}\tan(2x)\right)dy$$

b. $u = \frac{t^3r^6}{s^2}$

The differential of the function is given by:

$$du = f_t dt + f_r dr + f_s ds$$

But, $f_t = \frac{\partial}{\partial t} f(t,r,s) = \frac{\partial}{\partial t}\left(\frac{t^3r^6}{s^2}\right)$

$$= \frac{3t^2r^6}{s^2}$$

$f_r = \frac{\partial}{\partial r} f(t,r,s) = \frac{\partial}{\partial r}\left(\frac{t^3r^6}{s^2}\right),$

$$= \frac{6t^3r^5}{s^2}$$

and, $f_s = \frac{\partial}{\partial s} f(t,r,s) = \frac{\partial}{\partial s}\left(\frac{t^3r^6}{s^2}\right),$

$$= \frac{-2t^3r^6}{s^3}$$

Therefore,

$$du = \left(\frac{3t^2r^6}{s^2}\right)dt + \left(\frac{6t^3r^5}{s^2}\right)dr + \left(\frac{-2t^3r^6}{s^3}\right)ds$$

Note 4.2: Sometimes these total differentials are called simply the **differentials**.

Exercise 4.3

1. *Find the first order partial derivatives of the following functions*
   a. $f(x,y) = x^2 - xy + y^2$
   b. $f(x,y) = (x^2 - 1)(y + 2)$
   c. $f(x,y) = e^{xy} \ln y$
   d. $f(x,y) = (xy - 1)^2$
   e. $f(x,y) = \sqrt{x^2 + y^2}$
   f. $f(x,y) = \frac{x-y}{x+y}$
   g. $f(\alpha,\beta) = \sin\alpha \cos\beta$
   h. $f(t,w) = te^{w/t}$
   i. $f(r,s) = r\ln(r^2 + s^2)$
   j. $f(u,v) = e^v/(u + v^2)$
2. *Find f_x, f_y and f_z for the following functions*
   a. $f(x,y,z) = 1 + xy^2 - 2z^2$
   b. $f(x,y,z) = \ln(x + 2y + 3z)$
   c. $f(x,y,z) = x\sin y \cos z$
   d. $f(x,y,z) = \tanh(x + 2y + 3z)$
   e. $f(x,y,z) = yz\ln(xy)$
   f. $f(x,y,z) = x - \sqrt{y^2 + z^2}$
   g. $f(x,y,z) = (x^2 + y^2 + z^2)^{-1/2}$
   h. $f(x,y,z) = x(1 - \cos y) - z$
3. *Find the indicated partial derivatives of the following of the following functions*
   a. $f(x,y) = x^3y^5 + 2x^4y; f_{xx}, f_{xy}, f_{yx}, f_{yy}$
   b. $f(x,y) = e^{xe^y};\ f_{xx}, f_{xy}, f_{yx}, f_{yy}$
   c. $f(x,y) = 3xy^4 + x^3y^2; f_{xxy}, f_{yyy}$
   d. $f(r,s,t) = r\ln(rs^2t^3; f_{rss}, f_{rst})$
   e. $f(u,v,w) = \cos(4u + 3v + 2w); f_{uvw}, f_{vww}$
4. *Verify that the conclusion of Clairaut's theorem holds for the following*
   a. $h(x,y) = x\sin(x + 2y)$
   b. $h(x,y) = \ln(2x + 3y)$
   c. $h(x,y) = e^x + x\ln y + y\ln x$
   d. $h(x,y) = \ln\sqrt{x^2 + y^2}$
   e. $h(x,y) = x\sin y + y\sin x + xy$
   f. $h(x,y) = xye^y$
5. *Find the differential (total differential) of the following functions.*
   a. $z = y\cos xy$
   b. $z = \frac{y}{1+xyz}$
   c. $z = \alpha\beta^2\cos\gamma$
   d. $z = xye^{xz}$
   e. $z = x^3\ln(y^2)$
   f. $z = p^5q^3$

4.4. The chain rule

Overview

In this section we will discuss way of finding derivatives of functions of several variables using the idea of chain rule and we will differentiate different functions implicitly.

Section objective:

After the completion of this section, successful students be able to:

- Understand the idea of chain rule and implicit differentiation
- Differentiate functions implicitly

Case 1: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t), y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example 21: Compute $\frac{dz}{dt}$ for the function $z = xe^{xy}, x = t^2$ and $y = t^{-1}$

Solution: We apply the Chain Rule to find $\frac{dz}{dt}$ as follows:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial(xe^{xy})}{\partial x}.\frac{d(t^2)}{dt} + \frac{\partial(xe^{xy})}{\partial y}.\frac{d(t^{-1})}{dt}$$

$$= (e^{xy} + xye^{xy})(2t) + (x^2e^{xy})(-t^{-2})$$

$$= (2t)(e^{xy} + xye^{xy}) - t^{-2}x^2e^{xy}$$

$$= (2t)(e^t + te^t) - t^2e^t.$$

$$= 2te^t + t^2e^t$$

Therefore, $\frac{dz}{dt} = 2te^t + t^2e^t$.

Example 22: If $z = xy$, where $x = \cos t$ and $y = \sin t$, then find $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$.

Solution: We apply the Chain Rule to find $\frac{dz}{dt}$ as follows:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial(xy)}{\partial x} \cdot \frac{d(\cos t)}{dt} + \frac{\partial(xy)}{\partial y} \cdot \frac{d(\sin t)}{dt}$$

$$= (y)(-\sin t) + (x)(\cos t)$$

$$= (\sin t)(-\sin t) + (\cos t)(\cos t) \text{ (Since } x = \cos t, y = \sin t)$$

$$= -sin^2 t + cos^2 t$$

$$= \cos 2t.$$

Then value of the derivative at the point $t = \frac{\pi}{2}$ is

$$\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}} = \cos\left(2.\frac{\pi}{2}\right) = \cos\pi = -1.$$

Case 2: Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t), y = h(s, t)$ are both differentiable functions of s and t.Then,

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} \qquad \frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example 23: Find $\frac{dz}{ds}$ and $\frac{dz}{dt}$ for $z = e^{2r} \sin(3\theta), r = st - t^2, \theta = \sqrt{s^2 + t^2}$

Solution: Applying Case 2 of the Chain Rule, we get

$$\frac{dz}{ds} = \frac{\partial z}{\partial r}\frac{dr}{ds} + \frac{\partial z}{\partial \theta}\frac{d\theta}{ds}$$

$$= \frac{\partial(e^{2r} \sin(3\theta))}{\partial r} \cdot \frac{d(st - t^2)}{ds} + \frac{\partial(e^{2r} \sin(3\theta))}{\partial \theta} \cdot \frac{d(\sqrt{s^2+t^2})}{ds}$$

$$= (2e^{2r} \sin(3\theta))(t) + (3e^{2r} \cos(3\theta))\left(\frac{s}{\sqrt{s^2+t^2}}\right)$$

$$= t\left(2e^{2(st-t^2)} \sin\left(3\sqrt{s^2 + t^2}\right)\right) + \frac{3se^{2(st-t^2)} \cos\left(3\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}$$

Therefore, $\frac{dz}{ds} = t\left(2e^{2(st-t^2)} \sin\left(3\sqrt{s^2 + t^2}\right)\right) + \frac{3se^{2(st-t^2)} \cos\left(3\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial r}\frac{dr}{dt} + \frac{\partial z}{\partial \theta}\frac{d\theta}{dt}$$

$$= \frac{\partial(e^{2r} \sin(3\theta))}{\partial r} \cdot \frac{d(st - t^2)}{dt} + \frac{\partial(e^{2r} \sin(3\theta))}{\partial \theta} \cdot \frac{d\sqrt{s^2+t^2}}{dt}$$

$$= (2e^{2r} \sin(3\theta))(s - 2t) + (3e^{2r} \cos(3\theta))\left(\frac{t}{\sqrt{s^2+t^2}}\right)$$

$$= (s - 2t)\left(2e^{2(st-t^2)} \sin\left(3\sqrt{s^2 + t^2}\right)\right) + \frac{3te^{2(st-t^2)} \cos\left(3\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}.$$

Therefore, $\frac{dz}{ds} = t\left(2e^{2(st-t^2)} \sin\left(3\sqrt{s^2 + t^2}\right)\right) + \frac{3se^{2(st-t^2)} \cos\left(3\sqrt{s^2+t^2}\right)}{\sqrt{s^2+t^2}}$

Example 24: If $z = e^x \sin y$, where $x = st^2$ and $y = s^2t$, then find $\frac{dz}{ds}$ and $\frac{dz}{dt}$.

Solution: Applying Case 2 of the Chain Rule, we get

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

$$= \frac{\partial(e^x \sin y)}{\partial x}\cdot\frac{d(st^2)}{ds} + \frac{\partial(e^x \sin y)}{\partial y}\cdot\frac{d(s^2t)}{ds}$$

$$= (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$

$$= t^2 e^{st^2}\sin(s^2t) + 2ste^{st^2}\cos(s^2t)$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial(e^x \sin y)}{\partial x}\cdot\frac{d(st^2)}{dt} + \frac{\partial(e^x \sin y)}{\partial y}\cdot\frac{d(s^2t)}{dt}$$

$$= (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

$$= 2ste^{st^2}\sin(s^2t) + s^2e^{st^2}\cos(s^2t).$$

Note 4.3: Case 2 of the Chain Rule contains three types of variables: s and t are **independent** variables, x and y are called **intermediate** variables, and z is the **dependent** variable.

The general version of the chain rule

Suppose that f is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then, f is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{dz}{dt_i} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt_i} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt_i} + \cdots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt_i}$$

Example 25: If $f = x^4y + y^2z^3$, where $x = rse^t, y = rs^2e^{-t}$ and $z = r^2s\sin t$, then find the value of $\frac{\partial f}{\partial s}$ when $r = 2, s = 1$ and $t = 0$.

Solution: Using the general chain rule, we have

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}$$

$$= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t)$$

When, $r = 2, s = 1$ and $t = 0$, we have $x = 2, y = 2$ and $z = 0$, so

$$\frac{df}{ds} = (64)(2) + (16)(4) + (0)(0)$$
$$= 192.$$

1.4.1. Implicit differentiation

Theorem 4.3: (A Formula for Implicit Differentiation): Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Proof: Let $z = F(x, y)$ and since $F(x, y) = 0$, the derivative $\frac{dz}{dx}$ must be zero. Computing the derivative from the Chain Rule, we find

$$0 = \frac{dz}{dx} = \frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} \text{ (Since} F(x, y) = 0)$$
$$= F_x.1 + F_y.\frac{dy}{dx}$$

Now, solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Example 26: Find $\frac{dy}{dx}$ if $x^3 + y^3 = 6xy$.

Solution: The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

and $F_x = \frac{\partial}{\partial x}(x^3 + y^3 - 6xy) = 3x^2 - 6y, F_y = \frac{\partial}{\partial y}(x^3 + y^3 - 6xy) = 3y^2 - 6x$

Therefore,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x}$$
$$= -\frac{x^2 - 2y}{y^2 - 2x}$$

Example 27: Find $\frac{dy}{dx}$ for $3y^4 + x^7 = 5x$

Solution: The given equation can be written as

$$F(x, y) = 3y^4 + x^7 - 5x = 0$$

and $F_x = \frac{\partial}{\partial x}(3y^4 + x^7 - 5x) = 7x^6 - 5, F_y = \frac{\partial}{\partial y}(3y^4 + x^7 - 5x) = 12y^3$

Therefore,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{7x^6-5}{12y^3}$$

Thus, $\frac{dy}{dx} = -\frac{x^2-2y}{y^2-2x}$

Note 4.4: Now we suppose that is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. This means that $F\big(x, y, f(x, y)\big) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial z}\frac{dz}{dx} = 0$$

But, $\frac{dx}{dx} = 1$ and $\frac{dy}{dx} = 0$, so the above equation becomes:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{dz}{dx} = 0$$

If $\frac{\partial F}{\partial z} \neq 0$, we solve for $\frac{dz}{dx}$ and we obtain

$$\frac{dz}{dx} = -\frac{F_x}{F_z}$$

Using similar procedures, we obtain the formula for $\frac{dz}{dy}$ as

$$\frac{dz}{dy} = -\frac{F_y}{F_z}$$

Example 28: Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution: Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$, then

$$\frac{dz}{dx} = -\frac{F_x}{F_z} = -\frac{3x^2+6yz}{3z^2+6xy}$$

$$= -\frac{x^2+2yz}{z^2+2xy}$$

$$\frac{dz}{dy} = -\frac{F_y}{F_z} = -\frac{3y^2+6xz}{3z^2+6xy}$$

$$= -\frac{y^2+2xz}{z^2+2xy}$$

Example 29: Find $\frac{dz}{dx}$ and $\frac{dz}{dy}$ for $x^3z^2 - 5xy^5z = x^2 + y^3$

Solution: Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$, then

$$F_x = \frac{\partial}{\partial x}(x^3 + y^3 + z^3 + 6xyz - 1) = 3x^2 - 6yz$$

$$F_y = \frac{\partial}{\partial x}(x^3 + y^3 + z^3 + 6xyz - 1) = 3y^2 - 6xz$$

$$F_z = \frac{\partial}{\partial z}(x^3 + y^3 + z^3 + 6xyz - 1) = 6xy$$

Therefore, $\frac{dz}{dx} = -\frac{F_x}{F_z} = -\frac{3x^2-6yz}{6xy}$

and $\frac{dz}{dy} = -\frac{F_y}{F_z} = -\frac{3y^2-6xz}{6xy}$

Exercise 4.4

1. *Using the chain rule find* $\frac{df}{dt}$ *for the following*
   a. $f = x^2 + y^2 + xy, x = \sin t, y = e^t$
   b. $f = \sqrt{1 + x^2 + y^2}, x = \ln t, y = \cos t$
   c. $f = \ln\sqrt{x^2 + y^2 + z^2}, x = \sin t, y = \cos t, z = \tan t$
   d. $f = xe^{y/z}, x = t^2, y = 1 - t, z = 1 + 2t$
2. *Using the chain rule find* $\frac{df}{ds}$ *and* $\frac{df}{dt}$ *for the following*
   a. $f = \tan(u/v), u = 2s + 3t, v = 3s - 2t$
   b. $f = x^2y^3, x = s\cos t, y = s\sin t$
   c. $f = e^r\cos\theta, r = st, \theta = \sqrt{s^2 + t^2}$
   d. $f = xe^{y-z^2}, x = 2st, y = s - t, z = s + t$
   e. $f = \ln(x^2 + y^2 + z^2), x = s + 2t, y = 2s - t, z = 2st$
3. *Use the chain rule to find the indicated partial derivatives*
   a. $f = x^2 + xy^3, x = uv^2 + w^3, x = uv^2 + w^3, y = u + ve^w,\ \frac{df}{du}, \frac{df}{dv}$ *and* $\frac{df}{dw}$
   b. $f = \sqrt{r^2 + s^2}, r = y + x\cos t, s = x + y\sin t;\ \frac{df}{dx}, \frac{df}{dy}$ *and* $\frac{df}{dt}$
   c. $f = x^2 + yz, x = pr\cos t, y = pr\sin t, z = p + r; \frac{df}{dp}, \frac{df}{dr}$ *and* $\frac{df}{dt}$
4. *Find* $\frac{dy}{dx}$ *for the following and find the value of* $\frac{dy}{dx}$ *at the given point.*
   a. $x^3 - 2y^2 + xy = 0, at\ (1,1)$
   b. $xy + y^2 - 3x - 3 = 0, at\ (-1,1)$
   c. $x^2 + xy + y^2 - 7 = 0, at\ (1,2)$

d. $xe^y + \sin xy + y - \ln 2 = 0, at\ (0, \ln 2)$

5. *Find* $\frac{dz}{dx}$ *and* $\frac{dz}{dy}$ *for the following and find their values at the given point.*

   a. $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0, at\ (1, \ln 2, \ln 3)$

   b. $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0, at\ (2,3,6)$

   c. $z^3 - xy + yz + y^3 - 2 = 0, at\ (1,1,1)$

   d. $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0, at\ (\pi, \pi, \pi)$

4.5. Directional derivatives and gradients

Overview

In this section we study the definition of directional derivative and gradient and we will find directional derivatives and gradients of different functions

Section objective:

After the completion of this section, successful students be able to:

- Define directional derivative and gradients
- Find directional derivative and gradient

4.5.1. Directional derivative

Definition 4.10: The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $u = ai + bj$ is

$$D_u f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0, y_0)}{h}$$

if the limit exists.

Example 30: Find the derivative of

$$f(x, y) = x^2 + xy$$

at $P_0(1,2)$ in the direction of the unit vector $u = \left(1/\sqrt{2}\right)i + \left(1/\sqrt{2}\right)j$.

Solution:

$$D_u f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0, y_0)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(1+h\left(\frac{1}{\sqrt{2}}\right), 2+h\left(\frac{1}{\sqrt{2}}\right)\right) - f(1,2)}{h}$$

$$= \lim_{h\to 0} \frac{\left(1+\frac{h}{\sqrt{2}}\right)^2 + \left(1+\frac{h}{\sqrt{2}}\right)\left(\left(2+\frac{h}{\sqrt{2}}\right)\right) - (1^2+1.2)}{h}$$

$$= \lim_{h\to 0} \frac{\left(1+\frac{2h}{\sqrt{2}}+\frac{h^2}{2}\right)+\left(2+\frac{3h}{\sqrt{2}}+\frac{h^2}{2}\right)-3}{h}$$

$$= \lim_{h\to 0} \frac{\frac{5h}{\sqrt{2}}+h^2}{h}$$

$$= \lim_{h\to 0} \frac{5}{\sqrt{2}} + h = \frac{5}{\sqrt{2}} + 0$$

$$= \frac{5}{\sqrt{2}}$$

Therefore, the directional derivative of $f(x,y) = x^2 + xy$ at $P_0(1,2)$ in the direction $u = \left({}^{1}/_{\sqrt{2}}\right) i + \left({}^{1}/_{\sqrt{2}}\right) j$ is $\frac{5}{\sqrt{2}}$.

Theorem 4.4: If f is a differentiable function of x and y , then f has a directional derivative in the direction of any unit vector $u = ai + bj$ and

$$D_u f(x,y) = f_x(x,y)a + f_y(x,y)b$$

Proof: If we define a function g of the single variable by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of a derivative, we have

$$g'(0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

$$= D_u f(x_0, y_0) \qquad (1)$$

On the other hand, we can write $g(h) = f(x,y)$, where $x = x_0 + ha, y = y_0 + hb$, then

$$g'(h) = \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}$$

$$= f_x(x,y)a + f_y(x,y)b$$

If we now put $h = 0$, then $x = x_0, y = y_0$ and

$$g'(0) = f_x(x_0,y_0)a + f_y(x_0,y_0)b \qquad (2)$$

Now, combining equations (1) and (2), we get

$$D_u f(x_0, y_0) = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

Remark 4.6: If the unit vector u makes an angle θ with the positive x −axis, then we can write $u = \cos\theta\, i + \sin\theta\, j$ and the formula in theorem 4.4 becomes:

$$D_u f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta \qquad (3)$$

Example 31: Find the directional derivative $D_u(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

And u is the unit vector given by angle $\theta = {}^{\pi}/_{6}$ and what is $D_u f(1,2)$?

Solution: Here, $f_x(x, y) = 3x^2 - 3y$ and $f_y(x, y) = -3x + 8y$

Then, from formula (3), we have

$$D_u f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

$$= (3x^2 - 3y)\cos\frac{\pi}{6} + (-3x + 8y)\sin\frac{\pi}{6}$$

$$= (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\frac{1}{2}$$

$$= \frac{1}{2}\left[3\sqrt{3}x^2 - 3x + \left(8 - 3\sqrt{3}\right)y\right]$$

Therefore

$$D_u f(1,2) = \frac{1}{2}\left[3\sqrt{3}(1)^2 - 3.1 + \left(8 - 3\sqrt{3}\right)(2)\right]$$

$$= \frac{13 - 3\sqrt{3}}{2}$$

4.5.2. Gradient vector

Definition 4.11: If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by

$$\nabla f(x, y) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j = \langle f_x(x, y), f_y(x, y)\rangle$$

Example 32: Find $\nabla f(x, y)$ and $\nabla f(0,1)$, if $f(x, y) = sinx + e^{xy}$

Solution: $f_x(x, y) = \frac{\partial}{\partial x}(sinx + e^{xy})$

$$= \cos x + ye^{xy}$$

and $f_y(x,y) = \frac{\partial}{\partial y}(sinx + e^{xy})$

$= xe^{xy}$

Therefore

$$\nabla f(x,y) = \frac{\partial f}{\partial x}i + \frac{\partial f}{\partial y}j = (\cos x + ye^{xy})i + (xe^{xy})j$$

and $\nabla f(0,1) = 2i$.

- ✓ The relation between the directional derivative and the gradient vector is expressed as follows:

$$D_u f(x,y) = \nabla f(x,y).u$$

Example 33: Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of $v = 3i - 4j$.

Solution: The direction of v is the unit vector obtained by dividing v by its length. i.e.

$$u = \frac{v}{|v|} = \frac{v}{5} = \frac{3}{5}i - \frac{4}{5}j$$

The partial derivatives of f are everywhere continuous and these are:

$$f_x(x,y) = \frac{\partial}{\partial x}(xe^y + \cos(xy))$$

$$= e^y - \text{ysin}(xy)$$

and $f_x(2,0) = (e^y - \text{ysin}(xy))(2,0) = 1$

Similarly, $f_y(x,y) = \frac{\partial}{\partial y}(xe^y + \cos(xy))$

$$= xe^y - \text{xsin}(xy)$$

and $f_y(2,0) = (xe^y - \text{xsin}(xy))(2,0) = 2$

The gradient of f at $(2,0)$ is

$$\nabla f(2,0) = f_x(2,0)i + f_y(2,0)j$$

$$= i + 2j$$

The derivative of f at $(2,0)$ in the direction of $\boldsymbol{v}$ is therefore

$$D_u f(2,0) = \nabla f(2,0).u$$

$$= (i + 2j).\left(\frac{3}{5}i - \frac{4}{5}j\right)$$

$$= \frac{3}{5} - \frac{8}{5} = -1.$$

Functions of three variables

Definition 4.12: The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $u = ai + bj + ck$ is

$$D_u f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0, y_0, z_0)}{h}$$

if the limit exists.

Remark 4.7: If $f(x, y, z)$ is differentiable and $u = ai + bj + ck$, then

$$D_u f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

Definition 4.13: For a function of three variables, the **gradient vector**, denoted by ∇f is

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$$

Note 4.5: Similar to the function of two variables the relation between the directional derivative and the gradient vector is given as:

$$D_u f(x, y, z) = \nabla f(x, y, z).u$$

Example 34: If $f(x, y, z) = x \sin yz$, then

a. Find the gradient of f.
b. Find the directional derivative of f at $(1,3,0)$ in the direction of $v = i + 2j - k$.

Solution:

a. The gradient of f is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z)\rangle$$
$$= \langle \sin yz , xz \cos yz , xy \cos yz\rangle$$

b. At $(1,3,0)$ we have $\nabla f(1,3,0) = \langle 0,0,3\rangle = 3k$.

The unit vector in the direction of $v = i + 2j - k$ is

$$u = \frac{v}{|v|} = \frac{v}{\sqrt{6}} = \frac{1}{\sqrt{6}} i + \frac{2}{\sqrt{6}} j - \frac{1}{\sqrt{6}} k$$

Therefore,

$$D_u f(1,3,0) = \nabla f(1,3,0).u$$

$$= (3k).\left(\frac{1}{\sqrt{6}}i + \frac{2}{\sqrt{6}}j - \frac{1}{\sqrt{6}}k\right)$$
$$= \frac{-3}{\sqrt{6}}.$$

4.6. Tangent Planes and Tangent plane approximation

Overview

In this section we will discuss the idea of tangent plane and tangent plane approximation and we will find tangent planes and tangent plane approximations to different functions.

Section objective:

After the completion of this section, successful students be able to:

- Understand tangent planes and tangent plane approximations
- Find tangent plane and tangent plane approximation to the given function

4.6.1. Tangent plane

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first order partial derivatives, and let $P(x_0, y_0, z_o)$ be a point on S. Let $C_1$and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S.Then the point P lies on both C_1 and C_2 . Let $T_1$and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2 . (See Figure 10)

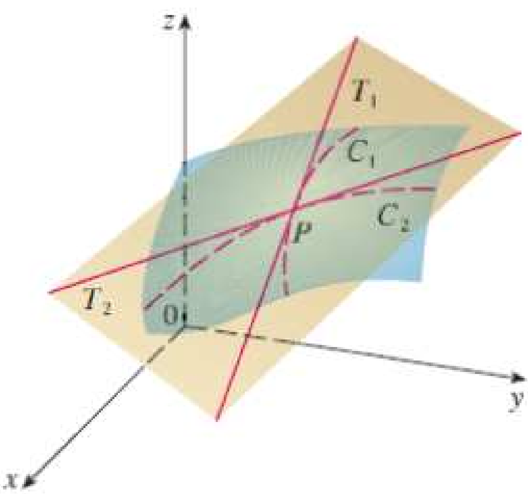

Figure 10

The tangent plane at P is the plane that most closely approximates the surface S near the point P. We know that the general equation of a plane which passes through (x_0, y_0, z_0) is given by,

$$a(x - x_0) + b(y - y_0) + c(z - z_0)$$

By dividing this equation by c and letting $A = -\frac{a}{c}$ and $B = -\frac{b}{c}$, we can write it in the form:

$$z - z_0 = A(x - x_0) + B(y - y_0) \qquad (1)$$

Let's first think about what happens if we hold y fixed, *i.e.* if we assume that $y = y_0$. In this case the equation of the tangent plane becomes,

$$z - z_0 = A(x - x_0), y = y_0$$

and we recognize these as the equations of a line with slope A and we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$.

Similarly, if $x = x_0$ in equation (1), then the equation becomes

$$z - z_0 = B(y - y_0), x = x_0$$

Which must represent the tangent line of T_2 and its slope is B= $f_y(x_0, y_0)$.

Definition 4.14: Suppose f has continuous partial derivatives. An equation of the tangent plane to thesurface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 35: Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1,1,3).

Solution: Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

$$f_x(1,1) = 4 \qquad f_y(1,1) = 2$$

Therefore, equation of the tangent plane at (1,1,3) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$\Rightarrow z - 3 = f_x(1,1)(x - 1) + f_y(1,1)(y - 1)$$

$$\Rightarrow z - 3 = 4(x - 1) + 2(y - 1)$$

$$\Rightarrow z = 4x + 2y - 3.$$

Thus, the equation is $z = 4x + 2y - 3$.

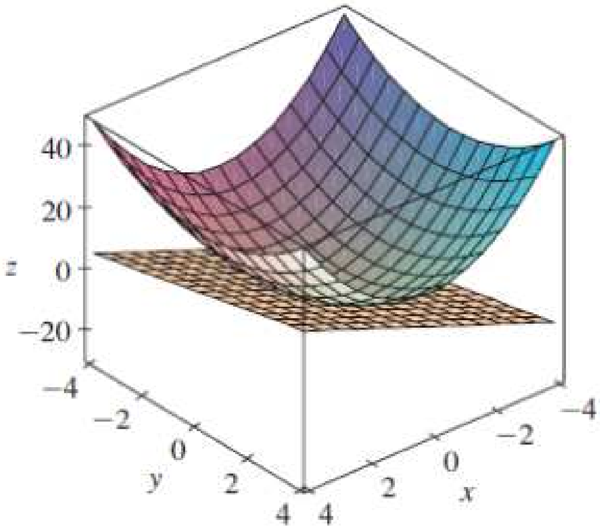

Figure 11

Example 36: Find the equation of the tangent plane to $z = \ln(2x + y)$ at the point $(-1,3,0)$.

Solution: Let $f(x, y) = \ln(2x + y)$. Then

$f_x(x, y) = \frac{2}{2x+y}$ $\qquad$ $f_y(x, y) = \frac{1}{2x+y}$

$f_x(-1,3) = 2$ $\qquad$ $f_y(-1,3) = 1$

Therefore, equation of the tangent plane at $(-1,3,0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$\Rightarrow z - 0 = f_x(-1,3)(x + 1) + f_y(-1,3)(y - 3)$$

$$\Rightarrow z = 2(x + 1) + (y - 3)$$

$$\Rightarrow z = 2x + y - 1.$$

Thus, the equation is $z = 2x + y - 1$.

4.6.2. Tangent plane approximation

One nice use of tangent planes is they give us a way to approximate a surface near a point. As long as we are near to the point (x_0, y_0) then the tangent plane should nearly approximate the function at that point.

In general, we know that an equation of the tangent plane to the graph of a function f of two variables at the point $(x_0, y_0, f(x_0, y_0))$ is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The linear function whose graph is this tangent plane, namely

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is called the **linearization** of f at (x_0, y_0) and the approximation

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (x_0, y_0).

Example 37: Find the tangent plane approximation to $z = 3 + \frac{x^2}{16} + \frac{y^2}{9}$ at $(-4,3)$.

Solution: Let $f(x, y) = 3 + \frac{x^2}{16} + \frac{y^2}{9}$. Then

$$f(-4,3) = 3 + 1 + 1 = 5$$

$$f_x(x, y) = \frac{x}{8} \qquad f_y(x, y) = \frac{2y}{9}$$

$$f_x(-4,3) = -\frac{1}{2} \qquad f_y(-4,3) = \frac{2}{3}$$

Then, the tangent plane approximation is

$$\begin{aligned} f(x, y) &\approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &= f(-4,3) + f_x(-4,3)(x + 4) + f_y(-4,3)(y - 2) \\ &= 5 + -\frac{1}{2}(x + 4) + \frac{2}{3}\ (\mathrm{y} - 4) \\ &= 2 - \frac{1}{2}x + \frac{2}{3}y \end{aligned}$$

Therefore, the tangent plane approximation is $L(x, y) = 2 - \frac{1}{2}x + \frac{2}{3}y$.

Example 38: Find the tangent plane approximation of $f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at the point (3,2).

Solution: Let $f(x, y) = x^2 - xy + \frac{1}{2}y^2 + 3$, then

$f(3,2) = 9 - 6 + 2 + 3 = 8$

$f_x(x, y) = 2x - y$ $\quad$ $f_y(x, y) = y - x$

$f_x(3,2) = 4$ $\quad$ $f_y(3,2) = -1$

Then, the tangent plane approximation is

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$
$$= f(3,2) + f_x(3,2)(x - 3) + f_y(3,2)(y - 2)$$
$$= 8 + 4(x - 3) + -1\ (y - 2)$$
$$= -2 + 4x - y$$

Therefore, the tangent plane approximation is $L(x, y) = 4x - y - 2$.

Exercise 4.5

1. *Find the directional derivatives of the following functions at P_0 in the direction of the given vectors.*
   a. $f(x, y) = 2xy - 3y^2, P_0 = (5,5), u = 4i + 3j$
   b. $f(x, y) = 2x^2 + y^2, P_0 = (-1,1), u = 3i - 4j$
   c. $f(x, y, z) = xy + yz + zx, P_0 = (1, -1,2), u = 3i + 6j - 2k$
   d. $h(x, y, z) = \cos xy + e^{yz} + \ln zx, P_0 = (1,0, {}^1/_2)$
   e. $h(x, y, z) = xe^y + ye^z + ze^x, P_0 = (0,0,0), u = 5i + j - 2k$
2. *For the following functions*
   h. Find the gradient vector.
   ii. Evaluate the gradient at the point P.
   a. $f(x, y) = \sin(2x + 3y), P = (-6,4)$
   b. $f(x, y) = y - x^2, P = (-1,0)$
   c. $f(x, y, z) = x^2 + y^2 - 2z^2 + z \ln x, P = (1,1,1)$
   d. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2} + \ln(xyz), P = (-1,2, -2)$
   e. $f(x, y, z) = xe^{2yz}, P = (3,0,2)$
3. *Find equations of the tangent plane to the given surface at the specified point*
   a. $2(x - 2)^2 + (y - 1)^2 + (z - 3)^2 = 10, P_0 = (3,3,5)$
   b. $x^2 + y^2 - 2xy - x + 3y - z = -4, P_0 = (2, -3,18)$

c. $xe^{y}\cos z - z = 1, P_0 = (1,0,0)$

d. $\cos\pi x - x^2y + e^{xz} + yz, P_0 = (0,1,2)$

e. $x^2 - 2y^2 + z^2 + yz = 2, P_0 = (2,1,-1)$

f. $yz - \ln(x+z) = 0, P_0 = (0,0,1)$

4. *Find the linearization of the function at each point*

a. $f(x,y) = x^2 + y^2 + 1\ at\ (0,0) and\ (1,1)$

b. $f(x,y) = e^{-xy}\cos y\ \ at\ (\pi,0)$

c. $f(x,y) = \sin(2x+3y)\ \ at\ (-3,2)$

d. $f(x,y) = 3x - 4y + 5\ at\ (0,0)\ and\ (1,1)$

e. $f(x,y,z) = xy + yz + xz\ at\ (1,1,1), (1,0,0)\ and\ (0,0,0)$

f. $f(x,y,z) = x^2 + y^2 + z^2\ at\ (1,1,1), (0,1,0)\ and\ (1,0,0)$

g. $f(x,y) = e^{x}\cos y\ at\ (0,0)\ and\ (0, {}^{\pi}/_{2})$

h. $f(x,y,z) = \sqrt{x^2+y^2+z^2}\ at\ (1,0,0), (1,1,0)\ and\ (1,2,2)$

4.7. Relative extrema of functions of two variables

Overview

In this section we define relative maximum value and relative minimum values of a function, absolute maximum and absolute minimum of functions, critical point and saddle point and we will find those defined terms for different functions

Section objective:

After the completion of this section, successful students be able to:

- Define relative and absolute maximum and minimum
- Find relative and absolute maximum and minimum values
- Define critical and saddle points
- Find critical and saddle points to given functions

Definition 4.15:

1. A function $f(x, y)$ has a relative minimum at the point (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) in some region around (a, b).
2. A function $f(x, y)$ has a relative maximum at the point (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in some region around (a, b).

Note 4.6: this definition does not imply that a relative minimum is the smallest value that the function will ever take. It only says that in some region around the point (a, b) the value of the function will always be larger than $f(a, b)$. Similarly, a relative maximum only says that around (a, b) the value of the function will always be smaller than $f(a, b)$.

- ✓ If the inequalities in the above definition hold for *all* points (x, y) in the domain of f, then f has an absolute maximum (or absolute minimum) at (a, b).
- ✓ The term relative extrema indicates both the relative minimum and relative maximum.

Definition 4.16: The point (a, b) is a critical point (or a stationary point) of $f(x, y)$ provided that one of the following is true:

1. $\nabla f(a, b) = 0$ (this is equivalent to saying that $f_x(a, b) = 0$ and $f_y(a, b) = 0$)
2. $f_x(a, b)$ and / or $f_y(a, b)$ does not exist.

Theorem 4.5: If the point (a, b) is a relative extrema of the function $f(x, y)$, then (a, b) is also a critical point of $f(x, y)$ and $\nabla f(a, b) = 0$.

Proof: Let $g(x) = f(x, b)$. If f has a local maximum (or minimum) at (a, b), then g has a local maximum (or minimum) at a and by Fermat's Theorem we have $g'(a) = 0$.

But, $g'(a) = f_x(a, b)$ and so $f_x(a, b) = 0$.

Similarly, let $h(y) = f(a, y)$. If f has a local maximum (or minimum) at (a, b), then h has a local maximum (or minimum) at b and by Fermat's Theorem we have $h'(b) = 0$.

But, $h'(b) = f_y(a, b)$ and so $f_y(a, b) = 0$.

Now, combining these two conditions together, we get $\nabla f(a, b) = 0$ and this indicates that (a, b) is a critical point of $f(x, y)$.

Example 39: Find the extreme values of $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

Solution: Let $f(x,y) = x^2 + y^2 - 2x - 6y + 14$. Then,

$$f_x(x,y) = 2x - 2 \quad \text{and} \quad f_y(x,y) = 2y - 6$$

Now, to find the critical points, we have

$$f_x(x,y) = 2x - 2 = 0 \quad \text{and} \quad f_y(x,y) = 2y - 6 = 0$$

That is $x = 1$ and $y = 3$ and therefore, the critical point is $(1,3)$.

By completing the square, we find that

$$f(x,y) = 4 + (x-1)^2 + (y-3)^2$$

Since $(x-1)^2 \geq 0$ and $(y-3)^2 \geq 0$, we have $f(x,y) \geq 4$ for all values of x and y.

Therefore, $f(1,3) = 4$ is a local minimum and in fact it is the absolute minimum.

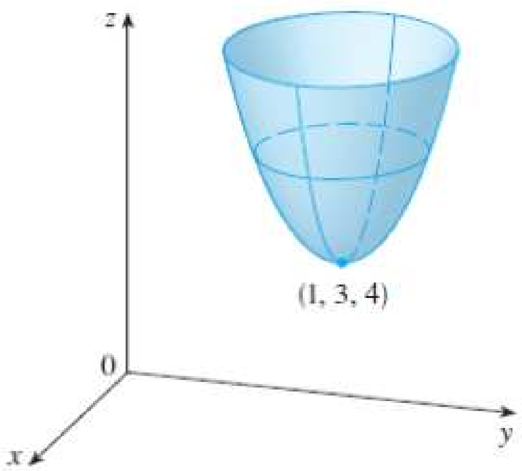

Figure 12 $\quad z = x^2 + y^2 - 2x - 6y + 14$

Example 40: Find the extreme values of $f(x,y) = y^2 - x^2$.

Solution: Let $f(x,y) = y^2 - x^2$. Then,

$$f_x(x,y) = -2x \quad \text{and} \quad f_y(x,y) = 2y$$

Now, to find the critical points, we have

$$f_x(x,y) = -2x = 0 \quad \text{and} \quad f_y(x,y) = 2y = 0$$

That is $x = 0$ and $y = 0$ and therefore, the critical point is $(0,0)$.

Notice that for points on the $x -$ axis we have $y = 0$, so $f(x,y) = -x^2 \leq 0$ (if $x \neq 0$). However, for points on the $y -$ axis we have $x = 0$, so $f(x,y) = y^2 \geq 0$ (if $y \neq 0$). Thus every disk with center $(0,0)$ contains points where f takes positive values as well as points where f takes negative values.

Therefore, $f(0,0) = 0$ can't be an extreme value for f, so f has no extreme value.

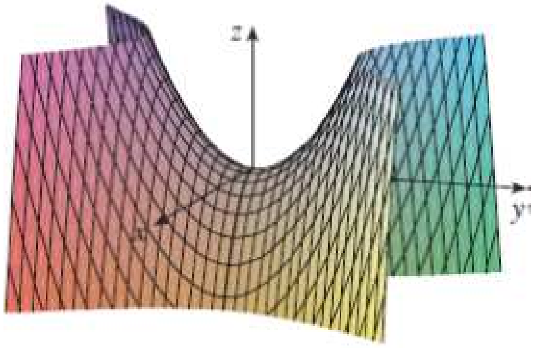

Figure 13 $z = y^2 - x^2$

Definition 4.17: A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface.

- Example 34 illustrates the fact that a function need not have a maximum or minimum value at a critical point. You can see that $f(0,0) = 0$ is a maximum in the direction of the x −axis but a minimum in the direction of the y −axis. Near the origin the graph has the shape of a saddle and so $(0,0)$ is a saddle point of f.

Second derivatives test: Suppose the second order partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ ((a, b) is a critical point of f). Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

a. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
b. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
c. If $D < 0$, then $f(a, b)$ is not a local minimum or local maximum.

Remark 4.8:

1. In case (c) the point (a, b) is called a saddle point of f and the graph of f crosses its tangent plane at (a, b).

2. If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.
3. To remember the formula for D, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - \left(f_{xy}\right)^2$$

Example 41: Find the local maximum and minimum values and saddle points of

$$f(x, y) = x^4 + y^4 - 4xy + 1.$$

Solution: We first find the critical points:

$$f_x = 4x^3 - 4y \quad \text{and} \quad f_y = 4y^3 - 4x$$

Setting these partial derivatives equal to 0, we obtain the equations

$$x^3 - y = 0 \quad \text{and} \quad y^3 - x = 0$$

To solve these equations we substitute $y = x^3$ from the first equation into the second one. This gives

$$\begin{aligned} 0 &= x^9 - x = x(x^8 - 1) \\ &= x(x^4 - 1)(x^4 + 1) \\ &= x(x^2 - 1)(x^2 + 1)(x^4 + 1) \end{aligned}$$

So there are three real roots:$x = 0,1,-1$. The three critical points are $(0,0), (-1,-1)$ and $(1,1)$.

Next we calculate the second partial derivatives and $D(x, y)$:

$$f_{xx} = 12x^2 - 4 \qquad f_{xy} = -4 \qquad f_{yy} = 12y^2 - 4$$

$$D(x, y) = f_{xx}f_{yy} - \left(f_{xy}\right)^2 = 144x^2y^2 - 16$$

and $D(0,0) = -16$, $D(-1,-1) = 128$ and $f_{xx}(-1,-1) = 12$ and $D(1,1) = 128$ and $f_{xx}(1,1) = 12$.

Since $D(0,0) = -16 < 0$, it follows from case (c) of the Second Derivatives Test that the origin is a saddle point; that is, f has no local maximum or minimum at $(0,0)$.

Since $D(1,1) = 128 > 0$ and $f_{xx}(1,1) = 12 > 0$, we see from case (a) of the test that $f(1,1) = -1$ is a local minimum. Similarly, we have $D(-1,-1) = 128 > 0$ and $f_{xx}(-1,-1) = 12 > 0$, so $f(-1,-1) = -1$ is also a local minimum.

The graph of f is shown in Figure 14 below.

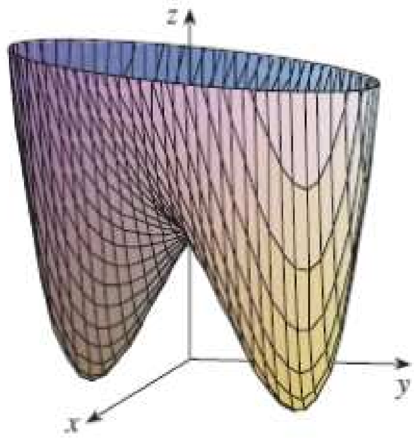

Figure 14 $\quad z = x^4 + y^4 - 4xy + 1$

Largest and smallest values of a function on a given set

Extreme Value Theorem for Functions of Two Variables: If f is continuous on a closed, bounded set D in $\mathbb{R}^3$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2).

Remark 4.9: To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value and the smallest of these values is the absolute minimum value.

- Absolute maximum is also called the largest value and absolute minimum is also called the smallest value.

Example 42: Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0, y = 0, y = 9 - x$.

Solution: Since f is differentiable, the only places where f can assume these values are points inside the triangle (Figure 15), where $f_x = f_y = 0$ and points on the boundary.

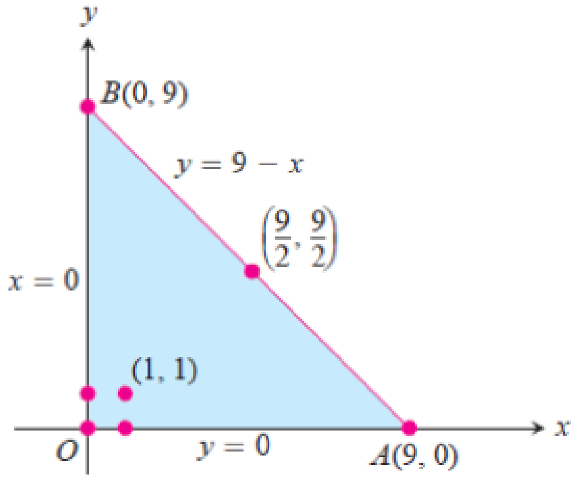

Figure 15 This triangular region is the domain of the function in Example 5.

a. **Interior points**. For these we have

$$f_x = 2 - 2x = 0 \quad \text{and} \quad f_y = 2 - 2y = 0$$

yielding the single point $(x, y) = (1,1)$ and the value of f at this point is

$$f(1,1) = 4$$

b. **Boundary points**. We take the triangle's one side at a time:

1. On the segment OA, $y = 0$ and the function becomes

$$f(x, y) = f(x, 0) = 2 + 2x - x^2$$

This function can be regarded as a function of x defined on the closed interval $0 \leq x \leq 9$ and its extreme values may occur at the end points

For $x = 0$ we have $f(0,0) = 2$

For $x = 9$ we have $f(9,0) = -61$

and at the interior points we have $f'(x, 0) = 2 - 2x = 0$. The only interior point where $f'(x, 0) = 0$ is $x = 1$, and

$$f(1,0) = 3.$$

2. On the segment OB, $x = 0$ and the function becomes

$$f(x, y) = f(0, y) = 2 + 2y - y^2$$

This function can be regarded as a function of y defined on the closed interval $0 \leq y \leq 9$ and its extreme values may occur at the end points

For $y = 0$ we have $f(0,0) = 2$

For $y = 9$ we have $f(0,9) = -61$

and at the interior points we have $f'(0,y) = 2 - 2y = 0$. The only interior point where $f'(0,y) = 0$ is $y = 1$, and

$$f(0,1) = 3.$$

3. On the segment AB, $y = 9 - x$ and we have already accounted for the values of f at the endpoints of *AB*, so we need only look at the interior points of *AB*.

$$f(x,y) = f(x, 9-x) = 2 + 2x + 2(9-x) - x^2 - (9-x)^2$$
$$= -61 + 18x - 2x^2$$

and $f'(x, 9-x) = 18 - 4x$

Setting $f'(x, 9-x) = 18 - 4x = 0$ gives

$$x = \frac{18}{4} = \frac{9}{2}$$

At this value of x, we have

$$y = 9 - \frac{9}{2} = \frac{9}{2} \quad \text{and} \quad f\left(\frac{9}{2}, \frac{9}{2}\right) = -\frac{41}{2}$$

Therefore, the absolute maximum value of f is 4 which attains at $(1,1)$ and the absolute minimum value of f is -61 which attains at (9,0) and (0,9).

Exercise 4.6

5. *Find the local maximum, local minimum and saddle point(s) of the following functions.*
   a. $f(x,y) = 9 - 2x + 4y - x^2 - 4y^2$
   b. $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$
   c. $f(x,y) = 3 + 2x + 2y - 2x^2 - 2xy - y^2$
   d. $f(x,y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$
   e. $f(x,y) = 4x^2 - 6xy + 5y^2 - 20x + 26y$
   f. $f(x,y) = 3x^2 + 6xy + 7y^2 - 2x + 4y$
   g. $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$
6. *Find the absolute maximum and absolute minimum values of the following functions on the given domains.*

a. $f(x,y) = x^2 + y^2$ *on the closed rectangular plate bounded by the lines* $x = 0, y = 0, y + 2x = 2$ *in the first quadrant.*

b. $f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$ *on the closed triangular plate bounded by the lines* $x = 0, y = 2, y = 2x$ *in the first quadrant.*

c. $f(x,y) = 4x + 6y - x^2 - y^2$ *on the closed region* $D = \{(x,y)|0 \leq x \leq 4, 0 \leq y \leq 5\}$

d. $f(x,y) = x^3 - 3x - y^3 + 12y$ *on the quadrilateral whose vertices are* $(-2,3), (2,3), (2,2)$ *and* $(-2,-2)$.

e. $f(x,y) = 3 + xy - x - 2y$ *on the closed triangular region with vertices* $(1,0), (5,0)$ *and* $(1,4)$.

4.8. Lagrange Multipliers

Extreme values under constraint conditions, Lagrange multiplier

Overview

In this section we study the idea of Lagrange multiplier and we study how can we find extreme values of a given function using the idea of Lagrange multiplier.

Section objective:

After the completion of this section, successful students be able to:

- Understand the idea of Lagrange multiplier
- Apply Lagrange multiplier to find extreme values for a given function

Find points of continuity for different functions

Consider the function f of two variables and then finding an extreme value of f subjected to a certain constraint (side condition) of the form $g(x, y) = c$, that is, an extreme value of f on the level curve $g(x, y) = c$ (rather than on the entire domain of f) . If f has an extreme value on the level curve at the point (x_0, y_0), then under certain conditions there exists a number λ such that

$$grad\ f(x_0, y_0 = \lambda\ .\ grad\ g(x_0, y_0)$$

Theorem 4.6: - Let f and g be differentiable at (x_0, y_0). Let C be the level curve $g(x, y) = \text{c}$ that contains (x_0, y_0). Assume that C is smooth, and that (x_0, y_0) is not an end point of the curve. If $grad\ g(x_0, y_0) \neq 0$, and if f has an extreme value on C at (x_0, y_0), then there is a number λ such that

$$grad\ f(x_0, y_0) = \lambda\ .\ grad\ g(x_0, y_0)\ . \quad \text{..........................(*)}$$

The number $\boldsymbol{\lambda}$ is called **Lagrange multiplier** for the functions f and g .

The equation given in (*) is equivalent to the pair of equations

$$f_x(x_0, y_0) = \lambda\ .\ g_x(x_0, y_0) \quad \text{and} \quad f_y(x_0, y_0) = \lambda\ .\ g_y(x_0, y_0)$$

Method of determining extreme values by means of Lagrange multiplier

Assuming f has an extreme value on the level curve $g(x, y) = c$ and $\nabla g \neq 0$

1. Solve the equations

Constraint: $g(x, y) = c$

$$grad\ f(x, y) = \lambda . grad\ g(x, y)$$

Or

$$\begin{cases} f_x(x, y) = \lambda . g_x(x, y) \\ f_y(x, y) = \lambda . g_y(x, y) \end{cases}$$

2. Evaluate the values of f at each point of (x, y) that result from step 1, and at each end point (if any) of the curve.
   - The largest of these values computed is the **maximum** value of f .
   - The smallest of these values computed is the **minimum** value of f

Note 4.7: Constraint is any limiting condition; in our case is limiting function.

Example 43: Let $f(x, y) = x^2 + 4y^3$. Find the extreme values of f on the ellipse $x^2 + 2y^2 = 1$.

Solution: Let $g(x, y) = x^2 + 2y^2$

The constraint is $g(x, y) = x^2 + 2y^2 = 1$

Then we need to find the point (x, y) such that the following conditioned is satisfied

$$grad\ f(x, y) = \lambda . grad\ g(x, y)$$

Or

$$\begin{cases} f_x(x, y) = \lambda . g_x(x, y) \\ f_y(x, y) = \lambda . g_y(x, y) \end{cases}$$

$f_x(x, y) = 2x$ and $f_y(x, y) = 12y^2$

$g_x(x, y) = 2x$ and $g_y(x, y) = 4y$

Then, $grad\ f(x, y) = (2x)i + (12y^2)j$ and $grad\ g(x, y) = (2x)i + (4y)j$

Then, by step1, the equation which we will use to find x and y becomes

Constraint: $x^2 + 2y^2 = 1 \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)$

$$2x\mathrm{i} + 12\mathrm{y}^2\mathrm{j} = \lambda. 2x\mathrm{i} + 4\mathrm{yj}$$

Or

$$\begin{cases} 2x = \lambda.2x \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2) \\ 12y^2 = \lambda.4y \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3) \end{cases}$$

From (2) we obtain either $x = 0$ or $\lambda = 1$

If $x = 0$, then from (1) we get $y = \frac{1}{\sqrt{2}}$ or $y = -\frac{1}{\sqrt{2}}$

If $\lambda = 1$, then (3) becomes $12y^2 = 4y$ which gives the result $y = 0$ or $y = \frac{1}{3}$.

If $y = 0$, then from (1) we get $x = 1$ or $x = -1$.

If $y = \frac{1}{3}$, from (1) we get $x = \frac{\sqrt{7}}{3}$ or $x = -\frac{\sqrt{7}}{3}$.

The possible points that f will have an extreme value are:

$(0, \frac{1}{\sqrt{2}})$, $(0, -\frac{1}{\sqrt{2}})$, $(1, 0)$,(-1, 0) , $(\frac{\sqrt{7}}{3}, \frac{1}{3})$ and $(-\frac{\sqrt{7}}{3}, \frac{1}{3})$

Find the functional values at each point and compare, and then the largest value indicates the maximum value of f and the smallest value indicates the minimum value of f.

Now $f\left(0, \frac{1}{\sqrt{2}}\right) = \sqrt{2}$, $f\left(0, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2}, f(1,0) = 1 = f(-1,0)$ and $f\left(\frac{\sqrt{7}}{3}, \frac{1}{3}\right) = \frac{25}{27} = f\left(-\frac{\sqrt{7}}{3}, \frac{1}{3}\right)$.

Since $f\left(0, \frac{1}{\sqrt{2}}\right) = \sqrt{2}$ is the largest and $f\left(0, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$ is the smallest, we conclude that the maximum value of f is $\sqrt{2}$ and occurs at $(0, \frac{1}{\sqrt{2}})$ and the minimum value is $-\sqrt{2}$ which occurs at $(0, -\frac{1}{\sqrt{2}})$.

The Lagrange Method for Functions of Three Variables

By an argument similar to functions of two variables, it is possible to show that if f has an extreme value at (x_0, y_0, z_0), then $grad\ f(x_0, y_0, z_0)$ and $grad\ g(x_0, y_0, z_0)$, if not 0, are both normal to the level surface $g(x, y, z) = c$ at (x_0, y_0, z_0) , and hence are parallel to each other. Thus, there is a number $\boldsymbol{\lambda}$ called Lagrange multiplier such that

$$grad\ f(x_0, y_0, z_0) = \lambda\ .grad\ g(x_0, y_0, z_0)$$

To find the extreme vales of f subjected to the constraint $g(x, y, z) = c$, follow the same steps as of a function of two variables :

Assume that f has an extreme value on the level surface $g(x, y, z) = c$ and $\nabla g \neq 0$.

Step1. Solve the equations

Constraint: $g(x,y,z) = c$

$grad\ f(x,y,z) = \lambda. grad\ g(x,y,z)$

Or

$$\begin{cases} f_x(x,y,z) = \lambda \,.\, g_x(x,y,z) \\ f_y(x,y,z) = \lambda \,.\, g_y(x,y,z) \\ f_z(x,y,z) = \lambda \,.\, g_z(x,y,z) \end{cases}$$

Step2. Evaluate the values of f at each point of (x,y,z) that results from step 1,

- The largest of these values computed is the maximum value of f .
- The smallest of these values computed is the minimum value of f.

Example 44: Let $f(x,y,z) = xyz$ for $x \geq 0$, $y \geq 0$ and $z \geq 0$. Find the maximum value of f subjected to the constraint $2x + 2y + z = 108$.

Solution: Let $g(x,y,z) = 2x + 2y + z$, then the constraint is $g(x,y,z) = 2x + 2y + z = 108$.

Find the first partial derivatives of f and g so as to help us in getting the gradient of the functions

That is, $g_x(x,y,z) = 2$, $g_y(x,y,z) = 2$ and $g_z(x,y,z) = 1$
$f_x(x,y,z) = yz, f_x(x,y,z) = xz$ and $f_z(x,y,z) = xy$,

Then, $grad\ f(x,y,z) = (yz)i + (xz)j + (xy)k$ and $grad\ g(x,y,z) = 2i + 2j + k$.

The equation which we use to find x, y and z become

Constraint: $2x + 2y + z = 108 \ldots\ldots\ldots\ldots\ldots\ldots..(1)$

$$grad\ f(x,y,z) = \lambda \,.\, grad\ g(x,y,z)$$

Or

$$\begin{cases} yz \ = 2\lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2) \\ \mathrm{xz} = 2\lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (3) \\ \mathrm{xy} = \lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4) \end{cases}$$

Type equation here.Then solving for λ in terms of x, y and z, we obtain

$$\lambda = \frac{yz}{2} = \frac{xz}{2} = xy \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (5)$$

Since $f(x, y, z) = 0$ if x, y or z is 0, and since obviously 0 is not the maximum value of f subjected to the constraint $2x + 2y + z = 108$, we can assume that x, y and z are different from 0. Then from (5), we obtain that $x = y$ and $z = 2y$. Substituting these values in (1) gives $y = 18$, $x = 18$, and $z = 36$.

Hence, the maximum value is $f(18{,}18{,}36) = 11{,}664$.

Example 45: A rectangular box without a lid is to be made from 12m^2 cardboard. Find the maximum volume of such a box.

Solution: Let x , y and z be the length, width and height, respectively, of the box in metres. Then we wish to maximize $\text{V} = xyz$ subjected to the constraint $g(x, y, z) = 2xz + 2yz + xy = 12$.

Using the method of Lagrange multipliers, we look for values of x , y , z and λ such that $\nabla V = \nabla g(x, y, z)$ and $g(x, y, z) = 12$. This gives the equation

$$V_x = \lambda \,.\, g_x, V_y = \lambda \,.\, g_y, V_z = \lambda . g_z \quad \text{and} \quad 2xz + 2yz + xy = 12$$

The equation which we use to find x , y and z become

Constraint: $2xz + 2yz + xy = 12 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$

$$\begin{cases} yz = \lambda.(2z + y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2) \\ xz = \lambda.(2z + x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3) \\ xy = \lambda.(2x + 2y) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4) \end{cases}$$

Clearly $\lambda \neq 0$,otherwise $yz = xz = xy = 0$, which contradicts to the constraint given in (1). Again x, y and $z \neq 0$, otherwise $\text{V} = 0$, which cannot be maximum value. Having these fore mentioned into consideration if we solve the equations, we obtain $x = 2, y = 2$ and $z = 1$ which gives maximum volume $\text{V} = 4\text{m}^3$.

Example 46: Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$ .Do the same on the disk $x^2 + y^2 \leq 1$.

Solution: Let $g(x, y) = x^2 + y^2$. The constraint is $g(x, y) = x^2 + y^2 = 1$.

Using Lagrange multipliers, we solve the equations

$\nabla f = \lambda . \nabla g$ and $g(x, y, z) = 1$. This gives the equations

$$f_x = \lambda . g_x, f_y = \lambda . g_y \quad \text{and} \quad x^2 + y^2 = 1$$

Then, the equations become

$$x^2 + y^2 = 1 \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

$$\begin{cases} 2x = 2x\lambda \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (2) \\ 4y = 2y\lambda \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (3) \end{cases}$$

From (2) we have $x = 0$ or $\lambda = 1$.

If $x = 0$ from (1), we obtain $y = \pm 1$.

If $\lambda = 1$ from (3), we have $y = 0$.

If $y = 0$ from (1), we have $x = \pm 1$.

Thus, the possible points that f will have an extreme value are: $(0,1), (0, -1), (1,0)$ and $(-1,0)$.

Evaluating the functional values: $f(0,1) = 2, f(0, -1) = 2, f(1,0) = 1$ and $f(-1,0) = 1$.

Thus, the maximum value of f on the circle $x^2 + y^2 = 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(\pm 1,0) = 1$.

To find the extreme values on the disk $x^2 + y^2 \leq 1$, we compare the values of f at the critical points with the values at the points on the boundary.

Since $f_x(x, y) = 2x$ and $f_y(x, y) = 4y$, the only critical point is (0,0).

The values on the boundary are: $f(0, \pm 1) = 2$ and $f(\pm 1,0) = 1$.

Comparing these, the maximum value of f on the disk $x^2 + y^2 \leq 1$ is $f(0, \pm 1) = 2$ and the minimum value is $f(0,0) = 0$.

Example 47: Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and furthest from the point $(3,1, -1)$.

Solution: The distance from a point (x, y, z) to the point $(3,1, -1)$ is

$$\mathrm{d} = \sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}$$

$$\Longrightarrow \mathrm{d}^2 = (x-3)^2 + (y-1)^2 + (z+1)^2$$

Let $f(x,y,z) = \mathrm{d}^2 = (x-3)^2 + (y-1)^2 + (z+1)^2$

The constraint is $g(x,y,z) = x^2 + y^2 + z^2 = 4$

Using Lagrange multipliers, we solve $\nabla f = \lambda\,.\nabla g$ and $g(x,y,z) = 4$

Then, the equations become

$$x^2 + y^2 + z^2 = 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

$$\begin{cases} 2(x-3) = 2x\lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(2) \\ 2(y-1) = 2y\lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3) \\ 2(z+1) = 2z\lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(4) \end{cases}$$

Solving these we obtain the points: $\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$ and $\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$

Clearly f has minimum value at $\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$ and maximum value at $\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$. Hence the closest point is $\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right)$ and the furthest point is $\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$.

Exercise 4.7

1. *Find the maximum value of the function* $f(x,y,z) = x + 2y + 2z$ *on the curve of intersection of the plane* $x - y + z = 1$ *and the cylinder* $x^2 + y^2 = 1$.
2. *Find the maximum and minimum values of the function* $f(x,y) = 4x + 6y$ *subject to* $x^2 + y^2 = 13$.
3. *Find the extreme values of* $f(x,y) = 2x^2 + 3y^2 - 4x - 5$ *on the disk* $x^2 + y^2 \leq 16$.
4. *Find the maximum and minimum values of the function* $f(x,y) = x^2y$ *subject to* $x^2 + 2y^2 = 6$.

Unit Summary:

- A function of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the domain of f and its range is the set of values that f takes on.
- A function of three variables f is a rule that assigns to each ordered triple (x, y, z) in a domain D a unique real number $f(x, y, z)$.
- If f is a function of two variables with domain D, then the graph of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y)\epsilon D$.
- The set of points in the plane where a function $f(x, y)$ has a constant value, $f(x, y) = k$ is a level curve of f and the set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = k$ is called a level surface of f.
- Let f be a function of two variables, then the limit of $f(x, y)$ as (x, y) approches (a, b) is L written as

  $$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

  If for $\varepsilon > 0$, there exists a corresponding $\delta > 0$ such that

  $$\text{If } 0 < \sqrt{(x-a)^2 + (y-b)^2}, then\ |f(x, y) - L| < \varepsilon$$

- A function $f(x, y)$ is continuous at the point (a, b) if
  1. f is defined at (a, b)
  2. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists
  3. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$
- The partial derivative of $f(x, y)$ with respect to x at the point (a, b) is given by

  $$f_x(a, b) = \left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}, \text{ if the limit exists}$$

  and the partial derivative of $f(x, y)$ with respect of y at the point (a, b) is given by

  $$f_y(a, b) = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}, \text{if the limit exists.}$$

Similarly, we define partial derivatives of functions of three and above variables. When we find partial derivative with respect to x we regard y as a constant and when we find with respect to y we regard x as a constant.

- The directional derivative of f at (x_0, y_0) in the direction of the unit vector $u = ai + bj$ is

$$D_u f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0, y_0)}{h}, \text{ if the limit exists}$$

And the gradient of $f(x, y)$ is the vector function ∇f defined by

$$\nabla f(x, y) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

And if f is a function of three variables with $u = ai + bj + ck$, then

$$D_u f(x_0, y_0, z_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0, y_{0,} z_0)}{h}, \text{ if the limit exists}$$

And the gradient of $f(x, y, z)$ is the vector function ∇f defined by

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

- A function $f(x, y)$ has a relative minimum at the point (a, b) if $f(a, b) \le f(x, y)$ for all (x, y) in some region around (a, b).
- A function $f(x, y)$ has a relative maximum at the point (a, b) if $f(a, b) \ge f(x, y)$ for all (x, y) in some region around (a, b).
- A function $f(x, y)$ has an absolute minimum at the point (a, b) if $f(a, b) \le f(x, y)$ for all (x, y) in the domain f.
- A function $f(x, y)$ has an absolute maximum at the point (a, b) if $f(a, b) \ge f(x, y)$ for all (x, y) in the domain of f.

Miscellaneous Exercises

1. Find the limit of the following functions

   a. $\lim_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$ c. $\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

   b. $\lim_{\substack{(x,y)\to(1,1)\\ x\neq y}} \frac{x^2-2xy+y^2}{x-y}$

2. Find the point of continuity for the following functions

   a. $f(x,y) = \frac{x^2+y^2}{x^2-3x+2}$ c. $f(x,y,z) = \sqrt{x^2+y^2-1}$

   b. $f(x,y) = \ln(x^2+y^2)$

3. Using the limit definition of the partial derivatives find the specified partial derivatives of the following at the given points

   a. $f(x,y) = 1 - x + y - 3x^2y$, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(1,2)$.

   b. $f(x,y) = 4 + 2x - 3y - xy^2$, $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at $(-2,0)$.

4. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the following functions

   a. $f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$

   b. $f(x,y) = cos^2(3x - y^2)$

5. Find all the second order partial derivatives of the following functions

   a. $f(x,y) = x^2y + \cos y + y\sin x$

   b. $f(x,y) = \tan^{-1}(y/x)$

6. Using the chain rule find the specified partial derivatives for the following at the given points.

   a. $w = \ln(x^2+y^2+z^2)$, $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$, $\frac{dw}{dt}$ at $t = 3$.

   b. $w = \ln(x^2+y^2+z^2)$, $x = ue^v \sin u$, $y = ue^v \cos u$, $z = ue^v$, $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ at $(u,v) = (-2,0)$.

7. Assume y is a differentiable function of x and z is a differentiable function of x and y, then find the specified implicit differentiation of the following at the given points.

a. $xe^y + \sin xy + y^2 - 7 = 0, \frac{dy}{dx}$ at $(0, \ln 2)$.

b. $xe^y + ye^z - 2 - 3\ln 2 = 0, \frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1, \ln 2, \ln 3)$.

8. Find the directional derivatives of the following functions at the given point in the given direction.

a. $f(x, y, z) = \tan^{-1}(y/x) + \sqrt{3}\sin^{-1}(xy/2)$ at (1,1), $u = 3i - 2j$.

b. $f(x, y, z) = x^2 + 2y^2 - 3z^2$ at (1,1,1), $u = i + j + k$.

9. Find the gradient of the following functions at the given points

a. $f(x, y, z) = e^{x+y}\cos z + (y + 1)\sin^{-1} x$ at $(0,0, \pi/6)$

b. $f(x, y, z) = 2z^3 - 3(x^2 + y^2)z + \tan^{-1}(xz)$ at (1,1,1).

10. Find the equation of the tangent line in the given point for the following.

a. $x^2 + 2xy - y^2 + z^2 = 7$ at the point $(1, -1, 3)$.

b. $x^3 + 3x^2y^2 + y^3 + 4xy - z^2 = 0$ at the point (1,1,3).

11. Find the tangent plane approximation of the following functions at the given points.

a. $f(x, y, z) = e^x + \cos(y + z)$ at the point $(0, \pi/4, \pi/4)$.

b. $f(x, y, z) = \tan^{-1}(xyz)$ at the point (1,1,1).

12. Find the local maximum, local minimum, critical points and saddle points of the following functions.

a. $f(x, y) = 5xy - 7x^2 + 3x - 6y + 2$

b. $f(x, y) = 2xy - x^2 - 2y^2 + 3x + 4$

13. Find the absolute maximum and absolute minimum of the following functions

a. $f(x, y) = x^2 - xy + y^2 + 1$ on the closed triangular plate in the first quadrant bounded by the lines $x = 0, y = 4, y = x$.

b. $f(x, y) = x^2 + xy + y^2 - 6x + 2$ on the rectangular plate $0 \le x \le 5, -3 \le y \le 0$.

14. Using Lagrange multiplier find the greatest and smallest values that the function $f(x, y) = xy$ takes on the ellipse $\frac{x^2}{8} + \frac{y^2}{2} = 1$.

15. Using Lagrange multiplier find the points on the curve $x^2 + xy + y^2 = 1$ in the xy plane nearest to and farthest from the origin.

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Chapter Five

Multiple Integrals

Introduction

In studying a real – world phenomenon, a quantity being investigated usually depends on two or more independent variables. So we need to extend the basic ideas of the calculus of functions of a single variable to functions of several variables. Although the calculus rules remain essentially the same, the calculus is even richer. The derivatives of functions of several variables are more varied and more interesting because of the different ways in which the variables can interact.

This chapter deals with the concept of functions of several variables, domains and ranges of functions of several variables, level curves and level surfaces of those functions and their graphs. In addition the concept of limit and continuity of functions of several variables and partial derivatives with their applications are discussed under this chapter.

We can use functions of several variables in different applications, for instance functions of two variables can be visualized by means of level curves, which connect points where the function takes on a given value. Atmospheric pressure at a given time is a function of longitude and latitude and is measured in millibars. Here the level curves are the isobars.

Unit Objectives:

On the completion of this unit, successful students be able to:

- Evaluate double and triple integrals
- Change rectangular coordinate systems to polar, cylindrical and spherical coordinate systems
- Apply different coordinate systems to evaluate multiple integrals
- Apply multiple integrals
- Understand the idea of iterated integrals

5.1. Double Integral

Overview

In this subtopic we will see the definition of double integral, the idea of iterated integrals in the double integral and we will do different examples on this topic.

Section objective:

After the completion of this section, successful students be able to:

- Define double integrals
- Understand the idea of iterated integrals
- Give examples on double integral

Recall that, we defined the definite integral of a single variable $\int_a^b f(x)dx$ as a limit of the Riemann sums $\sum_{k=1}^{n} f(x_k{}^*)\Delta x_k$.that is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k{}^*)\Delta x_k$$

Where, $a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b$ are points in a partition of the interval $[a, b]$ and $x_k{}^*$ is a representative point in the sub – intervals $[x_{k-1}, x_k]$.

We now apply the same idea to define a definite integral of two variables

$$\iint_R f(x, y)dA$$

Over the rectangular region $R = \{(x, y)|\ a \leq x \leq b, c \leq y \leq d\}$

Before, defining the double integral let's describe the following steps.

Step1: Partition the interval $a \leq x \leq b$ into m – subintervals and the interval $c \leq y \leq d$ into n - subintervals. Using these subdivisions, partition the rectangle R into $N = mn$ cells (sub – rectangles).

Step2: choose a representative point $(x_k{}^*, y_k{}^*)$ from each cell in the partition of the rectangle R and form the sum

$$\sum_{k=1}^{N} f\ (x_k{}^*, y_k{}^*)\Delta A_k$$

Where ΔA_k is the area of the k^{th} representative cell and this is called the Riemann sum of $f(x,y)$ with respect to the partition and cell representative $({x_k}^*, {y_k}^*)$.

Definition 5.1: If f is defined on a closed, bounded rectangular region R in the $xy-$ plane, then the double integral of f over R is defined by

$$\iint_R f(x,y)dA = \lim_{N\to\infty} \sum_{k=1}^{N} f({x_k}^*, {y_k}^*) = \lim_{m,n\to\infty} \sum_{k=1}^{m}\sum_{k=1}^{n} f({x_k}^*, {y_k}^*)$$

if the limit exists.

The precise meaning of the limit in the above definition is that for every number $\varepsilon > 0$ there is an integer N such that

$$\left| \iint_R f(x,y)dA - \lim_{m,n\to\infty} \sum_{k=1}^{m}\sum_{k=1}^{n} f({x_k}^*, {y_k}^*) \right| < \varepsilon$$

for all integers m and n greater than N and for any choice of sample points $({x_k}^*, {y_k}^*)$ in R_k.

Volume interpretation: the double integral is interpreted as volume. i.e., if $f(x,y) \geq 0$ on R and $f({x_k}^*, {y_k}^*)\Delta A_k$ is in the volume of the parallelepiped with height $f({x_k}^*, {y_k}^*)$ and base area ΔA_k, then the Riemann sum establishes the total volume under the surface. i.e.

$$V \approx \sum_{k=1}^{m}\sum_{k=1}^{n} f({x_k}^*, {y_k}^*) = \lim_{m,n\to\infty} \sum_{k=1}^{m}\sum_{k=1}^{n} f({x_k}^*, {y_k}^*)$$

Remark 5.1: If $f(x,y) \geq 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)dA$$

EXAMPLE 1: Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the upper right corner of each square R_{ij}. Sketch the solid and the approximating rectangular boxes.

Solution: The squares are shown in Figure 1.

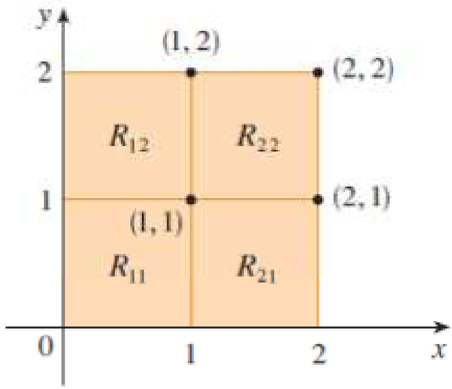

Figure 1

The paraboloid is the graph of $f(x,y) = 16 - x^2 - 2y^2$ and the area of each square is 1.

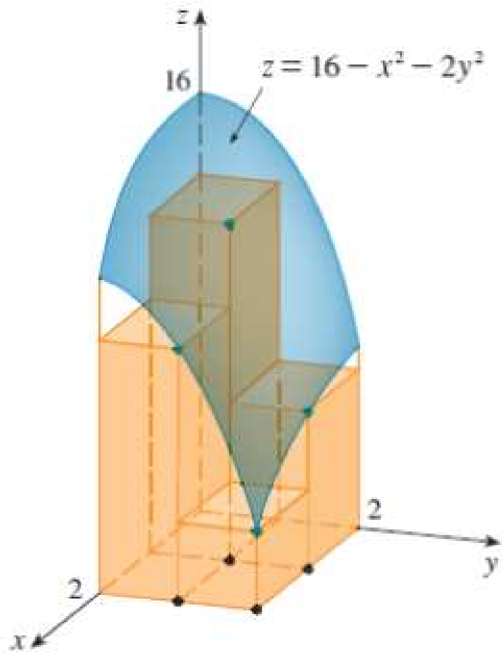

Figure 2

Approximating the volume by the Riemann sum with $m = n = 2$, we have

$$V \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, x_j)$$

$$= f(1,1)\Delta A + f(1,2)\Delta A + f(2,1)\Delta A + f(2,2)\Delta A$$

$$= 13(1) + 7(1) + 10(1) + 4(1) = 34$$

This is the volume of the approximating rectangular boxes shown in Figure 2.

Example 2: If $R = \{(x, y)|-1 \leq x \leq 1, -2 \leq y \leq 2\}$, evaluate the integral

$$\iint_R \sqrt{1-x^2}\, dA$$

Solution: It would be very difficult to evaluate this integral directly from Definition of the double integral but, because $\sqrt{1-x^2} \geq 0$, we can compute the integral by interpreting it as a volume. If $z = \sqrt{1-x^2}$, then $x^2 + y^2 = 1$ and $z \geq 0$, so the given double integral represents the volume of the solid *S* that lies below the circular cylinder $x^2 + y^2 = 1$ and above the rectangle *R.* (See Figure 3.)

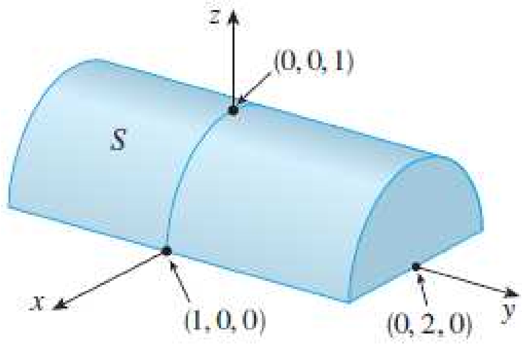

Figure 3

The volume of *S* is the area of a semicircle with radius 1 times the length of the cylinder. Thus,

$$\iint_R \sqrt{1-x^2} dA = \frac{1}{2}\pi(1)^2 \times 4 = 2\pi$$

5.1.1. Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are continuous then

1. Constant Multiple: $\iint_R cf(x,y)\, dA = c \iint_R f(x,y) dA$, (any number c)
2. Sum and Difference: $\iint_R f(x,y) \pm g(x,y) dA = \iint_R f(x,y)\, dA \pm \iint_R g(x,y) dA$
3. Domination:

   (a) $\iint_R f(x,y)\, dA \geq 0, if\ f(x,y) \geq 0\ on\ R$

   (b) $\iint_R f(x,y) dA \geq \iint_R g(x,y) dA\, , if\ f(x,y) \geq g(x,y)\ on\ R$
4. Additivity: $\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y) dA$

   if *R* is the union of two non – overlapping regions R_1 and R_2

5.1.2. Iterated integrals

Recall that it is usually difficult to evaluate single integrals directly from the definition of an integral, but the Fundamental Theorem of Calculus provides a much easier method. The evaluation of double integrals from first principles is even more difficult, but in this section we see how to express a double integral as an iterated integral, which can then be evaluated by calculating two single integrals.

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$. We use the notation $\int_c^d f(x, y)dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from c to d . This procedure is called partial integration with respect to y . (Notice its similarity to partial differentiation.) Now $\int_c^d f(x, y)dy$ is a number that depends on the value of x, so it defines a function of x:

$$A(x) = \int_c^d f(x, y)dy$$

If we now integrate the function A with respect to x from a to b , we get

$$\int_a^b A(x) = \int_a^b \left[\int_c^d f(x, y)dy\right] dx \qquad (1)$$

The integral on the right side of equation 1 is called an **iterated integral**. Usually the brackets are omitted. Thus

$$\int_a^b \int_c^d f(x, y)dy\, dx = \int_a^b \left[\int_c^d f(x, y)\, dy\right] dx \qquad (2)$$

means that we first integrate with respect to y from c to d and then with respect to x from a to b.

Similarly, the iterated integral

$$\int_c^d \int_a^b f(x, y)dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx\right] dy \qquad (3)$$

means that we first integrate with respect to x (holding y fixed) from $x = a\ to\ x =$b and then we integrate the resulting function of y with respect to y from $y = c\ to\ y = d$.

Example 3: Evaluate the iterated integrals of the following

(a) $\int_0^3 \int_1^2 x^2 y\, dy\, dx$ (b) $\int_1^2 \int_0^3 x^2 y\, dx\, dy$

Solution:

(a) Regarding x as a constant, we obtain

$$\int_1^2 x^2 y\, dy = \left[x^2 \frac{y^2}{2}\right]_1^2$$

$$= x^2\left(\frac{2^2}{2}\right) - x^2\left(\frac{1^2}{2}\right) = \frac{3}{2}x^2$$

Thus, the function A in the preceding discussion is given by $A(x) = \frac{3}{2}x^2$ in this example.

We now integrate this function of x from 0 to 3:

$$\int_0^3 \int_1^2 x^2 y\, dy\, dx = \int_0^3 \left[\int_1^2 x^2 y\, dy\right] dx$$

$$= \int_0^3 \frac{3}{2}x^2 dx = \left[\frac{x^3}{2}\right]_0^3$$

$$= \frac{27}{2}$$

(b) Here we first integrate with respect to x:

$$\int_1^2 \int_0^3 x^2 y\, dx\, dy = \int_1^2 \left[\int_0^3 x^2 y\, dx\right] dy$$

$$= \int_1^2 \left[\frac{x^3}{3} y\right]_0^3 dy$$

$$= \int_1^2 9y\, dy = \left[9\frac{y^2}{2}\right]_1^2 = \frac{27}{2}$$

Theorem 5.1 (Fubini's theorem): If is continuous on the rectangle

$R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x,y) dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y) dx\, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 4: Evaluate the double integral $\iint_R (x - 3y^2)dA$, where

$R = \{(x, y)|0 \leq x \leq 2, 1 \leq y \leq 2\}$.

Solution: Here $f(x, y) = x - 3y^2$ is continuous on R and applying Fubini's theorem and integrating with respect to y first:

$$\iint_R (x - 3y^2)dA = \int_0^2 \int_1^2 (x - 3y^2)dy\, dx$$

$$= \int_0^2 [xy - y^3]_1^2 dx$$

$$= \int_0^2 (x - 7)dx$$

$$= \left[\frac{x^2}{2} - 7x\right]_0^2 = -12$$

Applying Fubini's theorem and integrating with respect to x first:

$$\iint_R (x - 3y^2)dA = \int_1^2 \int_0^2 (x - 3y^2)dx\, dy$$

$$= \int_1^2 \left[\frac{x^2}{2} - 3xy^2\right]_0^2 dy$$

$$= \int_1^2 (2 - 6y^2)dy$$

$$= [2y - 2y^3]_1^2 = -12$$

Example 5: Evaluate $\iint_R y\sin(xy)dA$, where $R = \{(x, y)|1 \leq x \leq 2, 0 \leq y \leq \pi\}$

Solution: If we first integrate with respect x, we get

$$\iint_R y\sin(xy)dA = \int_0^\pi \int_1^2 y\sin(xy)dx\, dy = \int_0^\pi [-\cos(xy)]_1^2 dy$$

$$= \int_0^\pi (-\cos 2y + \cos y)dy$$

$$= \left[-\frac{1}{2}\sin 2y - \sin y\right]_0^\pi = 0$$

If we reverse the order of integration, we get

$$\iint_R \; y\sin(xy)dA = \int_1^2 \int_0^\pi y\sin(xy)dy\,dx$$

To evaluate the inner integral, we use integration by parts with

$$u = y \qquad\qquad dv = \sin(xy)$$

$$du = dy \qquad\qquad v = -\frac{\cos(xy)}{x}$$

and so $\int_0^\pi y\sin(xy)dy = \left[-\frac{y\cos(xy)}{x}\right]_0^\pi + \frac{1}{x}\int_0^\pi \cos(xy)dy$

$$= -\frac{\pi\cos\pi x}{x} + \frac{1}{x^2}[\sin(xy)]_0^\pi$$

$$= -\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}$$

If we now integrate the first term by parts with $u = 1/x$ and $dv = \pi\cos\pi x$, we get $du = \frac{dx}{x^2}, v = \sin\pi x$ and

$$\int \left(-\frac{\pi\cos\pi x}{x}\right)dx = -\frac{\sin\pi x}{x} - \int \frac{\sin\pi x}{x^2}dx$$

Therefore, $\int \left(-\frac{\pi\cos\pi x}{x} + \frac{\sin\pi x}{x^2}\right)dx = -\frac{\sin\pi x}{x}$

and so, $\int_1^2 \int_0^\pi y\sin(xy)dy\,dx = \left[-\frac{\sin\pi x}{x}\right]_1^2$

$$= -\frac{\sin 2\pi}{2} + \sin\pi = 0$$

Remark 5.2: $\iint_R$ f(x) h(y) $dA = \int_a^b f(x)dx \int_c^d h(y)dy$, where $R = [a,b]\times[c,d]$

Example 6: If $R = [0,\pi/2]\times[0,\pi/2]$, then

$$\iint_R \sin x\cos y dA = \int_0^{\pi/2}\sin x dx \int_0^{\pi/2}\cos y dy$$

$$= [-\cos x]_0^{\pi/2} [\sin y]_0^{\pi/2}$$

$$= 1.1 = 1$$

Exercise 5.1

1. *Calculate the following iterated integrals.*

   a. $\int_1^3 \int_0^1 (1+4xy)dx\,dy$

   b. $\int_0^2 \int_0^{\pi} r\sin^2\theta\, d\theta\, dr$

   c. $\int_0^1 \int_0^1 \sqrt{s+t}\, ds\, dt$

   d. $\int_0^1 \int_0^3 e^{x+3y} dx\, dy$

   e. $\int_0^1 \int_0^1 xy\sqrt{x^2+y^2} dy\, dx$

   f. $\int_0^1 \int_0^1 (u-v)^5\, du\, dv$

   g. $\int_1^4 \int_1^2 \left(\frac{x}{y} - \frac{y}{x}\right) dy\, dx$

   h. $\int_0^2 \int_0^1 (2x+y)^8 dx\, dy$

2. *Calculate the double integrals over the given regions*

   a. $\iint_R (6x^2y^3 - 5y^4)dA$, $R = \{(x,y)|0 \le x \le 3, 0 \le y \le 1\}$

   b. $\iint_R \cos(x+2y)dA$, $R = \left\{(x,y)\middle|0 \le x \le \pi, 0 \le y\, {}^{\pi}/_{2}\right\}$

   c. $\iint_R x\sin(x+y)dA$, $R = \left[0, {}^{\pi}/_{6}\right] \times \left[0, {}^{\pi}/_{3}\right]$

   d. $\iint_R \frac{1}{xy}\, dA$, *over the region* $R = \{(x,y)|1 \le x \le 2, 1 \le y \le 2\}$

   e. $\iint_R y\cos xy\, dA$, *over the region* $R = \{(x,y)|0 \le x \le \pi, 0 \le y \le 1\}$

3. *Find the volume of the following regions*

   a. *The solid that lies under the plane* $3x+2y+z=12$ *and above the rectangle* $R = \{(x,y)|0 \le x \le 1, -2 \le y \le 3\}$.

   b. *The solid that lies under the hyperbolic paraboloid* $z = 4 + x^2 - y^2$ *and above the square* $R = [-1,1] \times [0,2]$.

   c. *The solid lying under the rectangle paraboloid* $\frac{x^2}{4} + \frac{y^2}{9} + z = 1$ *and above the rectangle* $R = [-1,1] \times [-2,2]$.

   d. *The solid enclosed by the surface* $z = 1 + e^x \sin y$ *and the planes* $x = \pm 1, y = 0, y = \pi$, *and* $z = 0$.

   e. *The solid enclosed by the surface* $z = 2 + x^2 + (y-2)^2$ *and the planes* $x = \pm 1, y = 0, y = 4$, *and* $z = 1$.

5.2. Double Integrals over General Regions

Overview

In this subtopic we study the different regions in integrating double integrals and we will see different examples concerning on this.

Section objective:

After the completion of this section, successful students be able to:

- Understand the different regions which helps us to integrate double integrals
- Evaluate double integrals in those regions

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

Where $g_1$and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figures 4, 5 and 6.

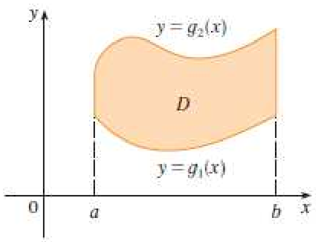

Figure 4

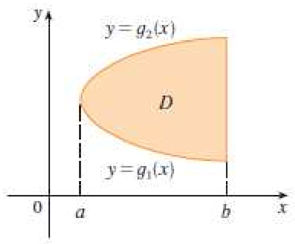

Figure 5

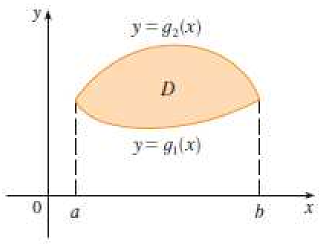

Figure 6

If f is continuous on a type I region D such that

$$D = \{(x, y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}, \text{ then}$$

$$\iint_D f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dy\, dx$$

We also consider plane regions of **type II**, which can be expressed as

$$D = \{(x, y)|c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

Where $h_1$and h_2 are continuous on $[c, d]$. One such region is given below.

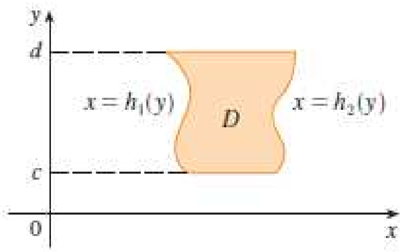

Figure 7

If f is continuous on a type II region D such that

$$D = \{(x, y)|c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}, \text{ then}$$

$$\iint_D f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)dx\, dy$$

Example 7: Evaluate $\iint_D (x + 2y)dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution: The parabolas intersect when $2x^2 = 1 + x^2$, that is $x^2 = 1$, so $x = \pm 1$. We note that the region, sketched in Figure 8, is a type I region but not a type II region

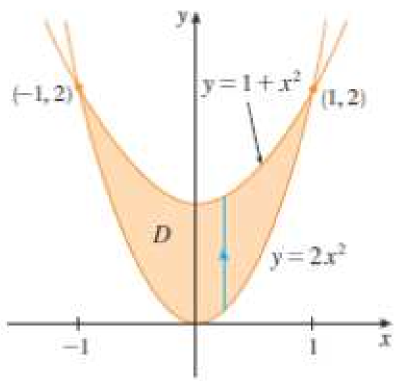

Figure 8

and we can write

$$D = \{(x, y)|-1 \leq x \leq 1, 2x^2 \leq y \leq 1 + x^2\}$$

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, then

$$\iint_R (x + 2y)dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)dy\, dx$$

$$= \int_{-1}^{1} [xy + y^2]_{2x^2}^{1+x^2}\, dx$$

$$= \int_{-1}^{1} [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2]\, dx$$

$$= \int_{-1}^{1} (-3x^4 - x^3 + x + 1)dx$$

$$= \left[-3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{1} = \frac{32}{15}$$

Example 8: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy −plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution:

Case I: consider the following figure

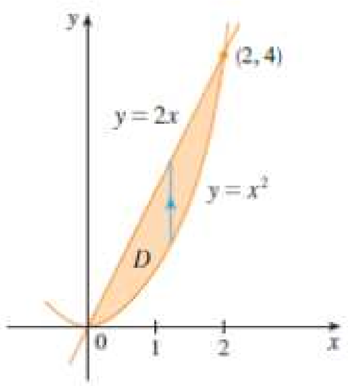

Figure 9

As we can see from the above graph D is a type I region and

$$D = \{(x, y)|0 \le x \le 2, x^2 \le y \le 2x\}$$

Therefore, the volume under $z = x^2 + y^2$ and above D is

$$V = \iint_D (x^2 + y^2)dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2)dy\, dx$$

$$= \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_{x^2}^{2x} dx$$

$$= \int_0^2 \left[x^2(2x) + \frac{(2x)^3}{3} - x^2x^2 - \frac{(x^2)^3}{3}\right] dx$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14x^3}{3}\right) dx$$

$$= \left[-\frac{x^7}{21} - \frac{x^5}{5} + \frac{7x^4}{6}\right]_0^2 = \frac{216}{35}$$

$$\Rightarrow V = \frac{216}{35}$$

Case II: consider the following figure

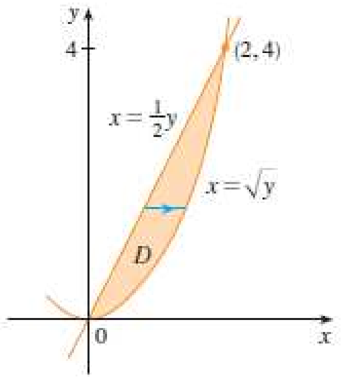

Figure 10

As we can see from the above graph D is a type II region and

$$D = \left\{(x, y) \middle| \, 0 \le y \le 4, \frac{1}{2}y \le x \le \sqrt{y}\right\}$$

Therefore, the other expression for the volume is

$$V = \iint_D (x^2 + y^2)dA = \int_0^4 \int_{\frac{1}{2}y}^{\sqrt{y}} (x^2 + y^2)dx\, dy$$

$$= \int_0^4 \left[\frac{x^3}{3} + y^2 x\right]_{\frac{1}{2}y}^{\sqrt{y}} dy$$

$$= \int_0^4 \left(\frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{y^3}{2}\right) dy$$

$$= \left[\frac{2y^{5/2}}{15} + \frac{2y^{7/2}}{7} - \frac{13y^4}{96}\right]_0^4 = \frac{216}{35}$$

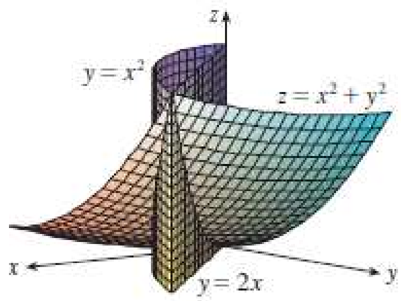

Figure 11

This is the graph of f together with D

Exercise 5.2

1. *evaluate the following iterated integrals*

a. $\int_0^4 \int_0^{\sqrt{y}} xy^2\ dx\, dy$

b. $\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta} dr\, d\theta$

c. $\int_0^1 \int_{x^2}^{x} (1 + 2y)\, dy\, dx$

d. $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\, dx\, dy$

e. $\int_0^1 \int_{2x}^{2} (x - y) dy\, dx$

f. $\int_0^2 \int_y^{2y} xy\, dx\, dy$

g. $\int_0^{\pi} \int_0^{x} x \sin y\, dy\, dx$

h. $\int_0^{3/2} \int_0^{9-4x^2} 16x\, dy\, dx$

2. *Evaluate the double integrals over the given regions.*

a. $\iint_D y^2\ dA, D = \{(x, y)| -1 \le y \le 1, -y - 2 \le x \le y\}$

b. $\iint_D y^2 e^{xy}\, dA, D = \{(x, y)|0 \le y \le 4, 0 \le x \le y\}$

c. $\iint_D x^3\, dA, D = \{(x, y)|1 \le x \le e, 0 \le y \le \ln x\}$

d. $\iint_D x\sqrt{y^2 - x^2}\, dA,\ D = \{(x, y)|0 \le y \le 1, 0 \le x \le y\}$

e. $\iint_D \frac{x}{y}\, dA$, *over the region* D *in the first quadrant bounded by the lines* $y = x, y = 2x, x = 1, x = 2.$

f. $\iint_D 3\cos t\, dA,\ D = \left\{(u, t)\middle|0 \le u \le \sec t, -\pi/_3 \le u \le \pi/_3\right\}$

3. *Find the volume of the following solid regions.*

   a. *Under the plane* $x + 2y - z = 0$ *and above the region bounded by* $y = x$ *and* $y = x^4$

   b. *Under the surface* $z = 2x + y^2$ *and above the region bounded by* $x = y^2$ *and* $x = y^3$

   c. *Bounded above by the cylinder* $z = x^2$ *and below by the region enclosed by the parabola* $y = 2 - x^2$ *and the line* $y = x$ *in the* xy *−plane.*

   d. *Under the plane* $z = x + 4$ *and below palane bounded by the parabola* $y = 4 - x^2$ *and the line* $y = 3x$.

5.3. Double Integrals in Polar Coordinates

Overview

In this section we will see the relation between the rectangular coordinate system and the polar coordinate system and we will study how double integrals are evaluated using polar coordinates.

Section objective:

After the completion of this section, successful students be able to:

- Justify the relation between the rectangular and polar coordinate systems
- Evaluate double integrals using polar coordinates

Suppose that we want to evaluate a double integral $\iint_R f(x, y)dA$, where R is one of the regions shown in Figs 12 & 13. In either case the description of R in terms of rectangular coordinates is rather complicated but R is easily described using polar coordinates.

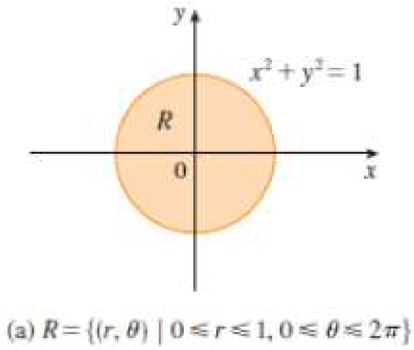

Figure 12

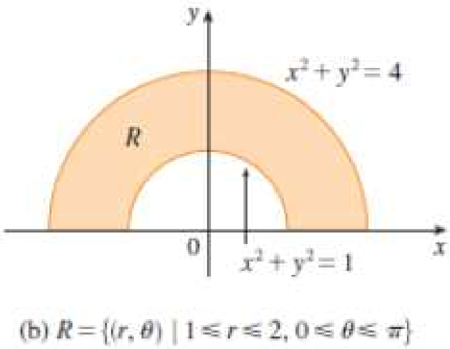

figure 13

Now consider the following figure

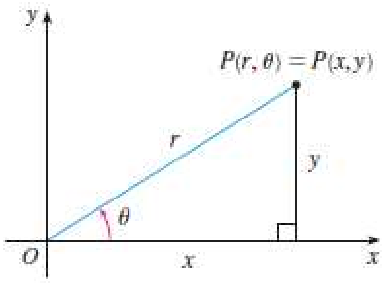

Figure 14

Recall from Figure 14 that the polar coordinates $(r, \theta)$of a point are related to the rectangular Coordinates (x, y) by the equations

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

The regions in Figure 12 & 13 are special cases of a **polar rectangle**

$$R = \{(r, \theta) |\ a \le r \le b, \alpha \le \theta \le \beta\}$$

which is shown in Figure 15. In order to compute the double integral $\iint_R f(x, y)dA$, where R is a polar rectangle, we divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = (b - a)/m$ and we divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = (\beta - \alpha)/n$. Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into the small polar rectangles shown in Figure 16.

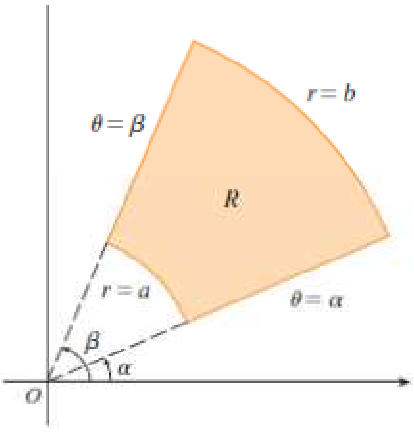

Figure 15

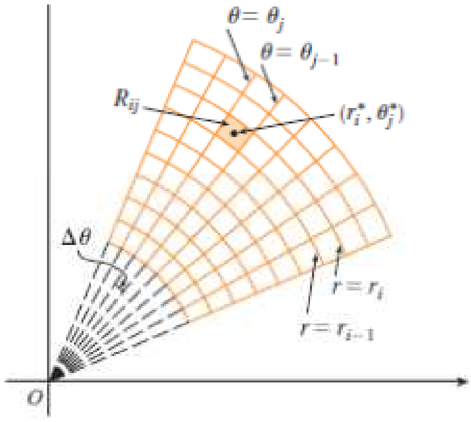

Figure 16

The “center” of the polar sub – rectangle

$$R_{ij} = \{(r,\theta) | r_{i-1} \leq r \leq r_i, \theta_{j-1} \leq \theta \leq \theta_j\}$$

has polar coordinates

$$r_i^* = \frac{1}{2}(r_{i-1} + r_i) \qquad \theta_j^* = \frac{1}{2}(\theta_{j-1} + \theta_j)$$

We compute the area of R_{ij} using the fact that the area of a sector of a circle with radius r and central angle θ is $\frac{1}{2}r^2\theta$. Subtracting the areas of two such sectors, each of which has central angle $\Delta\theta = \theta_j - \theta_{j-1}$, we find that the area of $R_{ij}$is

$$\Delta A_i = \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}(r_i^2 - r_{i-1}^2)\Delta\theta$$

$$= \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^*\Delta r\Delta\theta$$

$$\Rightarrow \Delta A_i = r_i^*\Delta r\Delta\theta$$

Now,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\,\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, r_i^*\Delta r\Delta\theta$$

Then, $\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, \Delta A_i$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, r_i^* \Delta r \Delta\theta$$

$$= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Definition 5.2: If f is continuous on a polar rectangle R given by $0 \le a \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example 9: Evaluate $\iint_R (3x + 4y^2)dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution: The region R can be described as

$$R = \{(x,y) | y \ge 0, 1 \le x^2 + y^2 \le 4\}$$

It is the half-ring shown in Figure 13 and in polar coordinates it is given by $1 \le r \le 2$, $0 \le \theta \le \pi$, therefore

$$\iint_R (3x + 4y^2)dA = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2 \sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3 \sin^2\theta)\, dr\, d\theta$$

$$= \int_0^\pi [r^3\cos\theta + r^4\sin^2\theta]_1^2\, d\theta$$

$$= \int_0^\pi (7\cos\theta + 15\sin^2\theta)\, d\theta$$

$$= \int_0^\pi \left[7\cos\theta + \frac{15}{2}(1 - \cos 2\theta)\right] d\theta$$

$$= \left[7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta\right]_0^\pi$$

$$= \frac{15\pi}{2}$$

Remark 5.3: If f is continuous on a polar region of the form

$$D = \{(r,\theta)|\alpha \le \theta \le \beta, h_1(\theta)\theta \le r \le h_2(\theta)\}$$

Then, $$\iint_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Example 10: Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy -plane, and inside the cylinder $x^2 + y^2 = 2x$.

Solution: The region D is given in the following figure

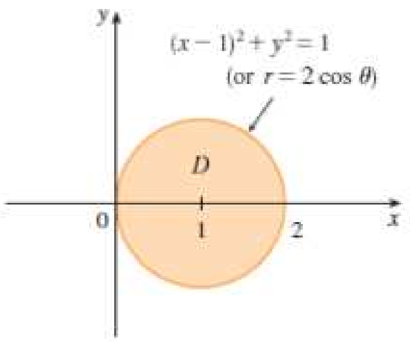

Figure 17

The solid lies above the disk D whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square, $(x-1)^2 + y^2 = 1$ (see figure 17 and 18).

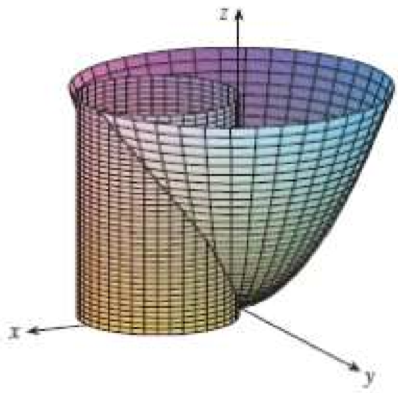

Figure 18

In polar coordinates we have $x^2 + y^2 = r^2$and $x = r\cos\theta$, so the boundary circle becomes $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$.

Thus the disk D is given by

$$D = \{(r,\theta)|-\pi/2 \le \theta \le \pi/2, 0 \le r \le 2\cos\theta\}$$

and therefore,

$$V = \iint_D (x^2+y^2)dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2 r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta = 8\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$= 8\int_0^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= 2\int_0^{\pi/2}\left[1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right] d\theta$$

$$= 2\left[\frac{3}{2}\theta + \sin 2\theta + \frac{1}{8}\sin 4\theta\right]_0^{\pi/2} = 2\left(\frac{3}{2}\right)\left(\frac{\pi}{2}\right) = \frac{3\pi}{2}$$

Exercise 5.3

1. *Evaluate the following integrals by converting them into polar coordinates*
   a. $\iint_D 2xy\,dA$, *D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.*
   b. $\iint_D e^{x^2+y^2} dA$, *D is the unit circle centered at the origin.*
2. *Determine the area of the region that lies inside $r = 3 + 2\sin\theta$ and out side $r = 2$.*
3. *Determine the volume of the region that lies under the sphere $x^2 + y^2 + z^2 = 9$, above the plane $z = 0$ and inside the cylinder $x^2 + y^2 = 5$.*
4. *Find the volume of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 16$.*

5.4. Applications of Double Integrals

Overview

In this section we will see the different applications of double integrals, such as areas and surface areas of different regions, mass and center of mass of different solid regions.

Section objective:

After the completion of this section, successful students be able to:

- Apply double integrals
- Exercise examples on the applications

We have so many applications of double integrals: computing volumes as we have seen before, finding areas of surfaces, physical applications such as computing mass, electric charge center of mass and moment of inertia are some of applications of double integrals.

5.4.1. Areas of Bounded Regions

Definition 5.3: The area of a closed, bounded plane region R is

$$A = \iint_R \; dA$$

Example 11: Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

Solution: We sketch the region as in the figure below

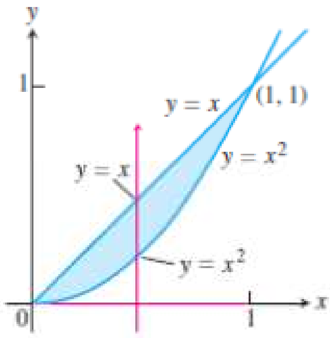

Figure 19

Then, we calculate the area of the region as follows

$$A = \iint_R \; dA = \int_0^1 \int_{y=x^2}^{y=x} dy\, dx$$

$$= \int_0^1 [y]_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$

5.4.2. Center of mass in double integral

Consider a lamina with variable density. Suppose the lamina occupies a region D and has density function $\rho(x, y)$. The moment of a particle about an axis is defined as the product of its mass and its directed distance from the axis. We divide D into small rectangles, then the mass of R_{ij} is approximately $\rho(x_{ij}{}^*, y_{ij}{}^*)\Delta A$, so we can approximate the moment of R_{ij} with respect to the -axis by

$$\left[\rho(x_{ij}{}^*, y_{ij}{}^*)\Delta A\right] y_{ij}{}^*$$

If we now add these quantities and take the limit as the number of subrectangles becomes large, we obtain the **moment** of the entire lamina **about the *x*-axis**:

$$M_x = \lim_{m,n\to\infty} \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} y_{ij}{}^* \rho(x_{ij}{}^*, y_{ij}{}^*)\Delta A$$

$$= \iint_D y\rho(x, y)dA$$

Similarly, the **moment** of the entire lamina **about the y-axis**:

$$M_y = \lim_{m,n\to\infty} \sum_{i=1}^{i=m} \sum_{j=1}^{j=n} x_{ij}{}^* \rho(x_{ij}{}^*, y_{ij}{}^*)\Delta A$$

$$= \iint_D x\rho(x, y)dA$$

We define the center of mass $(\bar{x}, \bar{y})$ so that $m\bar{x} = M_y$ and $m\bar{y} = M_y$

Definition 5.4: The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x, y)dA \qquad \text{and} \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x, y)dA$$

Where, the mass m is given by

$$m = \iint_D \rho(x, y)dA$$

Example 12: Find the mass and center of mass of a triangular lamina with vertices $(0,0), (1,0)\ and (0,2)$ if the density function is $\rho(x, y) = 1 + 3x + y$.

Solution: The triangle is shown in **Figure 20.** (Note that the equation of the upper boundary is $y = 2 - 2x$),

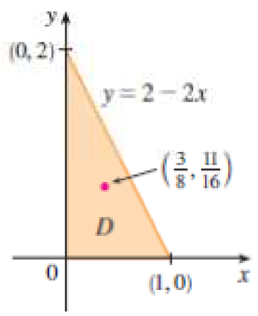

Figure 20

Then the mass of the lamina is

$$m = \iint_D \rho(x, y)dA = \int_0^1 \int_0^{2-2x} (1 + 3x + y)dy\,dx$$

$$= \int_0^1 \left[y + 3xy + \frac{y^2}{2}\right]_0^{2-2x} dx$$

$$= 4\int_0^1 (1 - x^2)dx = 4\left[x - \frac{x^3}{3}\right]_0^1 = \frac{8}{3}$$

and then the coordinates of the center of mass are

$$\bar{x} = \frac{1}{m}\iint_D x\rho(x, y)dA = \frac{3}{8}\int_0^1 \int_0^{2-2x} (x + 3x^2 + xy)dy\,dx$$

$$= \frac{3}{8}\int_0^1 \left[xy + 3x^2y + x\frac{y^2}{2}\right]_0^{2-2x} dx = \frac{3}{2}\int_0^1 (x - x^3)dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{8}$$

$$\bar{y} = \frac{1}{m}\iint_D y\rho(x, y)dA = \frac{3}{8}\int_0^1 \int_0^{2-2x} (y + 3xy + xy^2)dy\,dx$$

$$= \int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + x\frac{y^3}{3}\right]_0^{2-2x} dx = \frac{1}{4}\int_0^1 (7 - 9x - 3x^2 + 5x^3)dx$$

$$= \frac{1}{4}\left[7x - 9\frac{x^3}{3} - x^3 + 5\frac{x^4}{4}\right]_0^1 = \frac{11}{16}$$

Therefore, the center of mass is at the point $\left(\frac{3}{8}, \frac{11}{16}\right)$.

Example 5.13: A thin plate covers the triangular region bounded by the x −axis and the lines $x = 1$ and $y = 2x$ in the first quadrant. The plate's density at the point (x, y) is $\rho(x, y) = 6x + 6y + 6$. Find the plate's mass, first moments, and center of mass about the coordinate axes.

Solution: we sketch the plate as follows

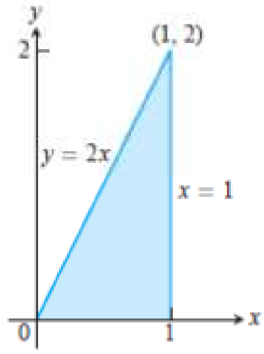

Figure 21

Then, the plate's mass is

$$m = \iint_D \rho(x, y)dA = \int_0^1 \int_0^{2x} (6x + 6y + 6)dy\, dx$$

$$= \int_0^1 \left[6xy + 3\frac{y^2}{2} + 6y\right]_0^{2x} dx$$

$$= \int_0^1 (24x^2 + 12x)dx$$

$$= [8x^3 + 6x^2]_0^1 = 14$$

The moment about the $x -$ axis is

$$M_x = \iint_D \ y\rho(x,y)dA = \int_0^1 \int_0^{2x} (6xy + 6y^2 + 6y)dy\, dx$$

$$= \int_0^1 [3xy^2 + 2y^3 + 3y^2]_0^{2x} dx$$

$$= \int_0^1 (28x^3 + 12x^2)dx$$

$$= [7x^4 + 4x^3]_0^1 = 11.$$

and similarly, the moment about the $y-$ axis is

$$M_y = \iint_D \ x\rho(x,y)dA = \int_0^1 \int_0^{2x} (6x^2 + 6xy + 6x)dy\, dx$$

$$= \int_0^1 [6x^2y + 3xy^2 + 6xy]_0^{2x} = 10.$$

The coordinates of the center of mass are therefore,

$$\bar{x} = \frac{M_y}{m} = \frac{10}{14} = \frac{5}{7} \quad \text{and} \quad \bar{y} = \frac{M_x}{m} = \frac{11}{14}$$

5.4.3. Surface area

Definition 5.5: Let R be a vertically or horizontally simple region and let f have continuous partial derivatives on R. then, the surface area S of the graph of f on R is defined by

$$S = \iint_R \sqrt{{f_x}^2 + {f_y}^2 + 1}$$

Example 14: Find the surface area of the paraboloid $z = 1 + x^2 + y^2$ that lies above the unit circle.

Solution: Here, the region R: $x^2 + y^2 \leq 1$. That is

$$R = \{(r,\theta)|0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$$

and $f(x,y) = z = 1 + x^2 + y^2$

Then, $f_x = 2x$ and $f_y = 2y$.

Therefore, $S = \iint_R \left(\sqrt{f_x{}^2 + f_y{}^2 + 1}\right) dA = \iint_R \left(\sqrt{(2x)^2 + (2y)^2 + 1}\right) dA$

$$= \iint_R \sqrt{4(x^2 + y^2) + 1}\, dA$$

$$= \int_0^{2\pi} \int_0^1 \left(\sqrt{r^2 + 1}\right) r dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \sqrt{u} \frac{du}{8r}\, r\, d\theta = \frac{1}{8} \int_0^{2\pi} \int_0^1 \sqrt{u}\, du\, d\theta$$

$$= \frac{1}{8} \int_0^{2\pi} \frac{2}{3} \left[(4r^2 + 1)^{\frac{3}{2}}\right]_0^1 d\theta = \frac{1}{12} \int_0^{2\pi} \left(5^{\frac{3}{2}} - 1\right) d\theta$$

$$= \frac{1}{12} \left[\left(5^{\frac{3}{2}} - 1\right) \theta\right]_0^{2\pi}$$

$$= \frac{\pi}{6}\left(5\sqrt{5} - 1\right)$$

Hence, the surface area is, $S = \frac{\pi}{6}\left(5\sqrt{5} - 1\right)$

Exercise 5.4

1. *Sketch the region bounded by the given lines and curves and then calculate the area of this region for the following.*
   a. *The coordinate axes and the line* $x + y = 2$
   b. *The lines* $x = 0, y = 2x$ *and* $y = 4$
   c. *The parabolas* $x = y^2$ *and* $x = 2y - y^2$
   d. *The parabolas* $x = y^2 - 1$ *and* $x = 2y^2 - 2$
   e. *The parabola* $x = -y^2$ *and* $y = x + 2$
2. *Find the mass and center of mass of the lamina that occupies the region* D *and has the given density function* ρ.
   a. $D = \{(x, y) |\ 0 \le x \le 2, -1 \le y \le 1\};\ \rho(x, y) = xy^2$
   b. $D = \{(x, y) |\ 0 \le x \le a, 0 \le y \le b\}; \rho(x, y) = cxy$

c. $D = \{(x,y) \mid 0 \le y \le \sin\left({}^{\pi x}/_{L}\right), 0 \le x \le L\}; \rho(x,y) = y$

d. D *is bounded by* $y = \sqrt{x}, y = 0$ *and* $x = 1;\ \rho(x,y) = x$

e. D *is bounded by* $y -$ *axis and the lines* $y = x$ *and* $y = 2 - x; \rho(x,y) = 6x + 3y + 3$

f. D *is bounded by the parabolas* $y = x^2$ *and* $x = y^2; \rho(x,y) = \sqrt{x}$

g. D *is bounded by the lines* $x = 6$ *and* $y = 1$ *in the first quadrant;* $\rho(x,y) = x + y + 1$

5.5. Triple integrals

Overview

In this subtopic we will see the definition of triple integral, the idea of iterated integrals in the triple integral and we will study the different regions which help us to evaluate triple integrals.

Section objective:

After the completion of this section, successful students be able to:

- Define triple integrals
- Understand the different types of regions to integrate triple integrals

Let's first deal with the simplest case where is f defined on a rectangular box:

$$B = \{(x,y,z) \mid a \le x \le b, c \le y \le d, r \le z \le s\}$$

The first step is to divide B into sub-boxes. We do this by dividing the interval $[a,b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c,d]$ into m subintervals of width Δy , and dividing $[r,s]$ into n subintervals of width Δz . The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

which are shown in Figure 22. Each sub-box has volume, $\Delta V = \Delta x \Delta y \Delta z$

Then we form the **triple Riemann sum**

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}{}^*, y_{ijk}{}^*, z_{ijk}{}^*)$$

Where, the sample point $(x_{ijk}{}^*, y_{ijk}{}^*, z_{ijk}{}^*)$ is in B_{ijk}.

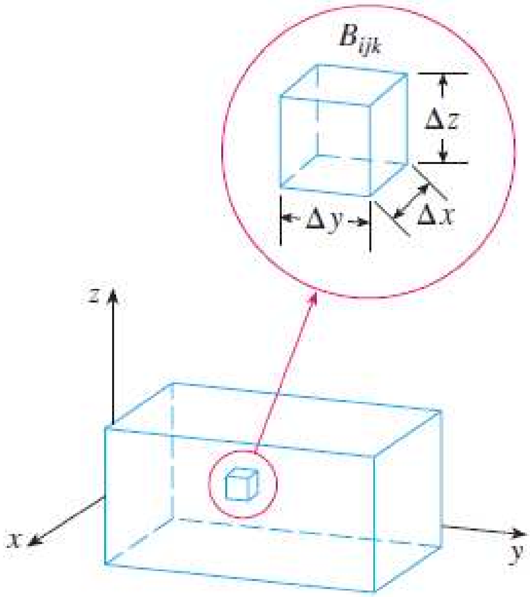

Figure 22

Definition 5.6: The triple integral of f over the region or box B is

$$\iiint_B f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}{}^*, y_{ijk}{}^*, z_{ijk}{}^*)\,\Delta V$$

If the limit exists

Theorem 5.2 (Fubini's theorem for triple integrals): If f is continuous on the rectangular box

$B = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dx\,dy\,dz$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to x (keeping y and z fixed), then we integrate with respect to y (keeping

z fixed), and finally we integrate with respect to z. Similar to the double integral we can interchange the variables of integration, but the result is the same in all cases.

Example 15: Evaluate the triple integral $\iiint_B xyz^2 dV$, where B is the rectangular box given by

$$B = \{(x,y,z)|0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$$

Solution: We could use any of the six possible orders of integration. If we choose to integrate with respect to x, then y , and then z, we obtain

$$\iiint_B xyz^2 dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 dx\, dy\, dz$$

$$= \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2}\right]_0^1 dydz$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2} dy\, dz$$

$$= \int_0^3 \left[\frac{y^2z^2}{4}\right]_{-1}^2 dz$$

$$= \int_0^3 \frac{3z^2}{4} dz = \left[\frac{z^3}{4}\right]_0^3 = \frac{27}{4}$$

Now we define the **triple integral over a general bounded region *E*** in three dimensional space. We restrict our attention to continuous functions and to certain simple types of regions. A solid region is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is

$$E = \{(x,y,z)|(x,y) \epsilon D, u_1(x,y) \le z \le u_2(x,y)\}$$

where D is the projection of E onto the xy −plane as shown in Figure 23. Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x,y)$, while the lower boundary is the surface $z = u_1(x,y)$.

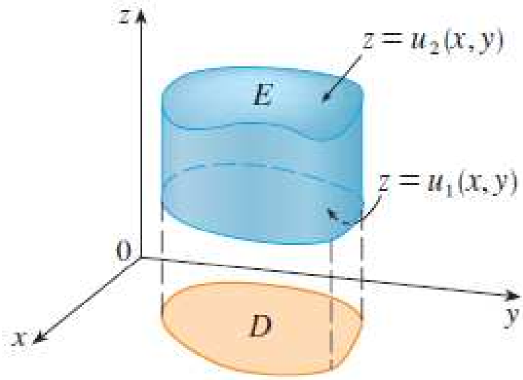

Figure 23 A type 1 solid region

Then, the triple integration over this region is defined as follows

$$\iiint_E f(x,y,z)dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\right] dA$$

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure 24)

Then,

$$E = \{(x,y,z)|a \le x \le b, g_1(x) \le y \le g_2(x), u_1(x,y) \le z \le u_2(x,y)\} \text{ and}$$

$$\iiint_E f(x,y,z)dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)dz\,dy\,dx$$

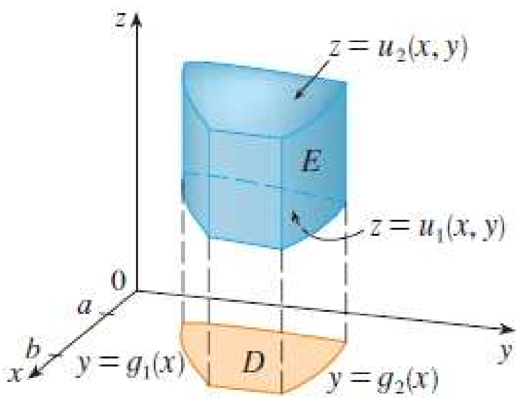

Figure 24 A type 1 solid region where the projection D is a type I plane region

On the other hand if D is a type II plane region (as in Figure 25),

Then,

$$E = \{(x, y, z) | c \leq y \leq d, h_1(y) \leq x \leq h_2(x), u_1(x, y) \leq z \leq u_2(x, y)\}$$

and the triple integral becomes

$$\iiint_E f(x, y, z) dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) dz\, dx\, dy$$

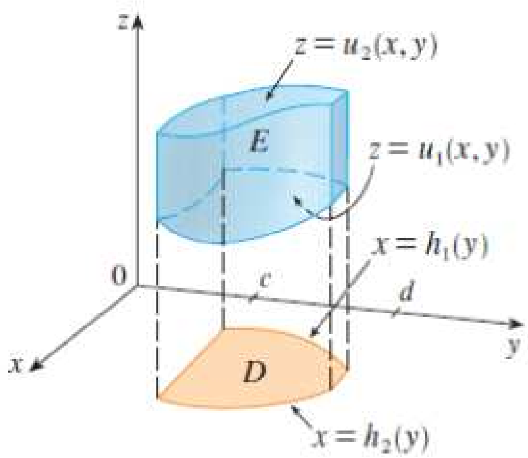

Figure 25 A type 1 solid region with a type II projection

Example 16: Evaluate $\iiint_E z\, dV$, where E is the solid tetrahedron bounded by the four Planes $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

Solution: First consider the following two figures

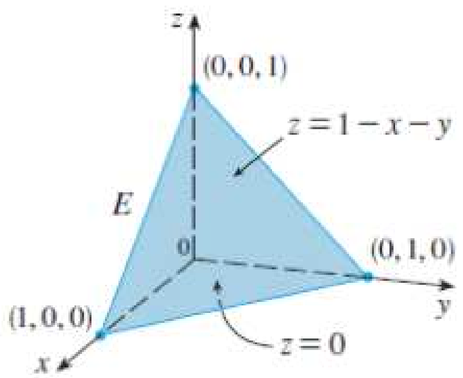

Figure 26

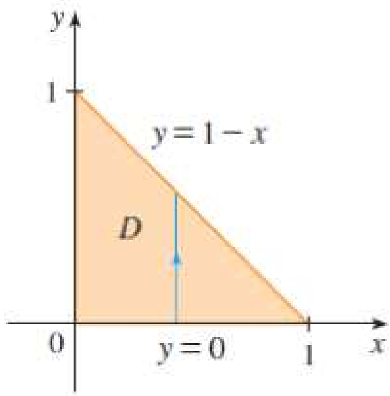

Figure 27

When we set up a triple integral first we have to draw *two* diagrams: one of the solid region E (see Figure 26) and one of its projection D on the xy −plane (see Figure 27). The lower boundary of the tetrahedron is the plane $z = 0$ and the upper is the plane $x + y + z = 1$ or $z = 1 - x - y$, we use $u_1(x, y) = 0$ and $u_2(x, y) = 1 - x - y$. The planes $x + y + z = 1$ and $z = 0$ intersect in the line $x + y = 1$ or $y = 1 - x$ in the xy −plane and therefore,

$$E = \{(x, y, z)|0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y\}$$

So, $$\iiint_E z\, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\, dz\, dy\, dx = \int_0^1 \int_0^{1-x} \left[\frac{z^2}{2}\right]_0^{1-x-y} dydx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} (1 - x - y)^2 dy\, dx$$

$$= \frac{1}{2}\int_0^1 \left[-\frac{(1-x-y)^3}{3}\right]_0^{1-x} dx$$

$$= \frac{1}{6}\int_0^1 (1 - x)^3 dx = \frac{1}{6}\left[-\frac{(1-x)^4}{4}\right]_0^1 = \frac{1}{24}$$

Therefore, $\iiint_E z\, dV = \frac{1}{24}$

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z)|(y, z)\epsilon D, u_1(y, z) \le x \le u_2(y, z)\}$$

Where, D is the projection of E onto the yz −plane (figure 28) and the back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$ and the integral becomes

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z)dx\right] dA$$

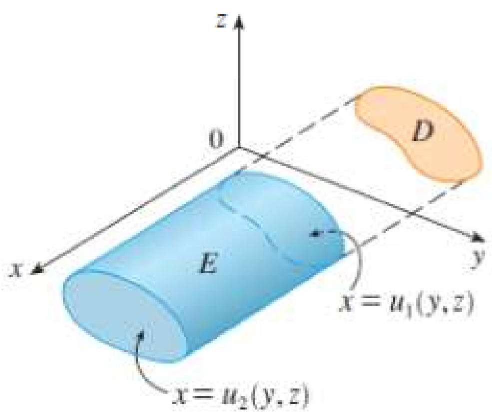

Figure 28 A type 2 region

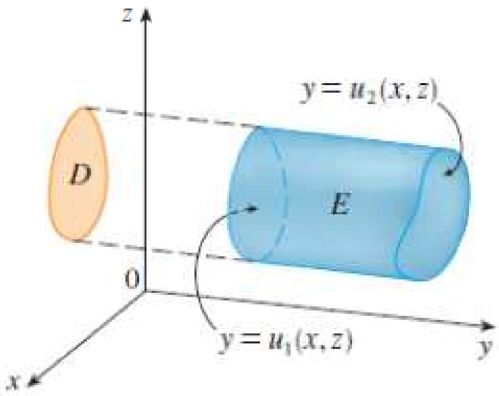

Figure 29 A type 3 region

Finally, a **type 3** region is of the form

$$E = \{(x, y, z)|(x, z)\epsilon D, u_1(x, z) \le y \le u_2(x, z)\}$$

Where, D is the projection of E onto the xz −plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface (see Figure 29). For this type of region we have

$$\iiint_E f(x, y, z)dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z)dy\right] dA$$

Remark 5.4: In the type 2 and type 3 there may be two possible expressions for the integral depending on whether D is a type I or type II plane region.

Example 17: Evaluate $\iiint_E \sqrt{x^2 + z^2}dV$, where is E the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

Solution: The solid E is shown in Figure 30. Consider E as a type 3 region. As such, its projection D_3 onto the xz −plane is the disk $x^2 + z^2 \le 4$ as shown in figure 31.

Then the left boundary of E is the paraboloid $y = x^2 + y^2$ and the right boundary is the plane $y = 4$, so taking $u_1(x,z) = x^2 + z^2$ and $u_2(x,z) = 4$.

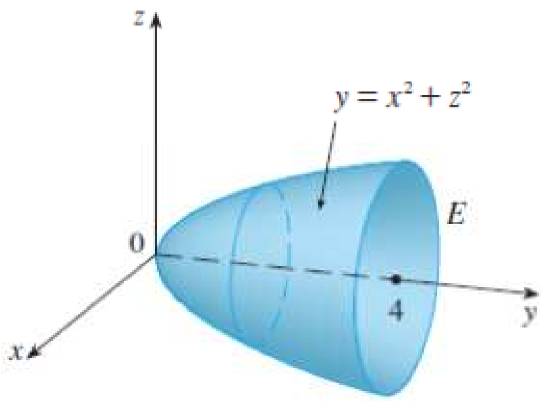

Figure 30 The region E

Then,

$$\iiint_E \sqrt{x^2+z^2}dV = \iint_D \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}dy\right] dA$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - z^2) \sqrt{x^2+z^2}dz\, dx$$

To evaluate the integral it is easier to convert to polar coordinates in the xz- plane, $x = r\cos\theta, z = r\sin\theta$.so,

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - x^2 - z^2) \sqrt{x^2+z^2}dz\, dx = \int_0^{2\pi} \int_0^2 (4 - r^2) r\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^2 (4r^2 - r^4) dr = 2\pi \left[\frac{4r^3}{3} - \frac{r^5}{5}\right]_0^2 = \frac{128\pi}{15}$$

$$\Rightarrow \iiint_E \sqrt{x^2+z^2}dV = \frac{128\pi}{15}$$

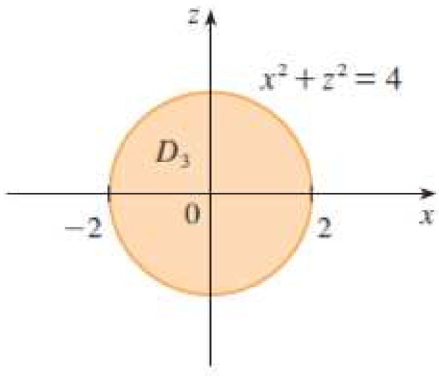

Figure 31 Projection on xz −plane

Exercise 5.5

1. *evaluate the iterated integrals of the following*
   a. $\int_0^1 \int_0^1 \int_0^1 (x^2 + y^2 + z^2)\, dz\, dy\, dx$
   b. $\int_0^{\pi} \int_0^{\pi} \int_0^{\pi} \cos(u + v + w)\, du\, dv\, dw$
   c. $\int_1^e \int_1^e \int_1^e \ln r \ln s \ln t\, dt\, dr\, ds$
   d. $\int_0^1 \int_0^z \int_0^{x+z} 6xz\, dy\, dx\, dz$
   e. $\int_0^{\pi/2} \int_0^y \int_0^x \cos(x + y + z)\, dz\, dx\, dy$
   f. $\int_0^1 \int_0^{1-x^2} \int_3^{4-x^2-y} x\, dz\, dy\, dx$
   g. $\int_0^1 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_0^{2x+y} dz\, dx\, dy$
2. *Evaluate the following triple integrals over the given region* E.
   a. $\iiint_E 2x\, dV$, $E = \{(x, y, z) | 0 \le y \le 2, 0 \le x \le \sqrt{4 - y^2}, 0 \le z \le y\}$.
   b. $\iiint_E z\, dV$, E *is bounded by the paraboloid* $x = 4x^2 + 4y^2$ *and the plane* $x = 4$.
   c. $\iiint_E y\, dV$, E *is bounded by the planes* $x = 0, y = 0, z = 0$ *and* $2x + 2y + z = 4$.
   d. $\iiint_E x^2 e^y\, dV$, E *is bounded by the parabolic cylinder* $z = 1 - y^2$ *and the planes* $z = 0, x = 1$ *and* $x = -1$.

e. $\iiint_E z\, dV$, *E is bounded by the cylinder* $y^2+z^2=9$ *and the planes* $x=0, y=3x$ *and* $z=0$ *in the first octant.*

5.6. Triple integrals in cylindrical and spherical coordinates

Overview

In this subtopic we will study the relations between rectangular and cylindrical coordinate systems, the rectangular and spherical coordinate systems; we study how can we evaluate triple integrals using cylindrical and spherical coordinates.

Section objective:

After the completion of this section, successful students be able to:

- Justify the relations between the coordinate systems
- Evaluate triple integrals using cylindrical and spherical coordinate systems

5.6.1. Triple integrals in cylindrical coordinates

In the **cylindrical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (r,θ,z) where $(r,\theta)$is the polar representation of the projection of P onto the xy −plane and z is the directed distance from the xy −plane to P.

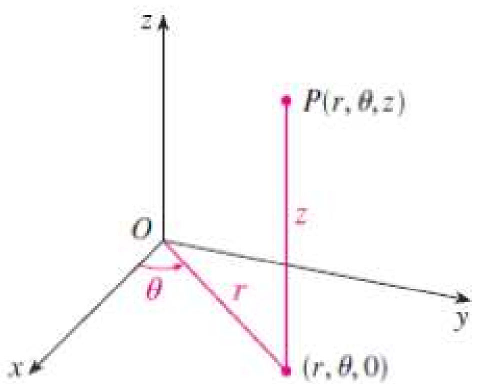

Figure 32 The cylindrical coordinates of a point

To convert from cylindrical to rectangular coordinates, we use the equations

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

But, to convert from rectangular to cylindrical coordinates, we use

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

Example 18: (a) Plot the point with cylindrical coordinates $(2,2\pi/3,1)$ and find its rectangular coordinates.

(b) Find cylindrical coordinates of the point with rectangular coordinates $(3,-3,-7)$.

Solution: (a) The point with cylindrical coordinates $(2,2\pi/3,1)$ is plotted in Figure 33.

Then, the rectangular coordinates are

$$x = r\cos\theta = 2\cos\frac{2\pi}{3} = 2\left(\frac{-1}{2}\right) = -1$$

$$y = r\sin\theta = 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$z = 1$$

Hence, the point in rectangular coordinate is $\left(-1,\sqrt{3},1\right)$.

(b) The cylindrical coordinates are

$$r^2 = x^2 + y^2 \ \Rightarrow r = \sqrt{x^2 + y^2}$$

$$\Rightarrow r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = \frac{y}{x} = \frac{-3}{3} = -1, \text{ so } \theta = \frac{7\pi}{4} + 2n\pi$$

$$z = -7$$

Therefore, the points in cylindrical coordinates are infinitely many by taking different values of n. For instance, $\left(3\sqrt{2},\frac{7\pi}{4},-7\right)$, $\left(3\sqrt{2},-\frac{\pi}{4},-7\right)$ and so on.

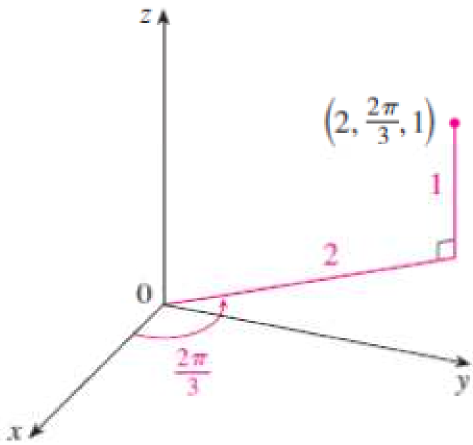

Figure 33

Suppose that E is a type 1 region whose projection D on the xy −plane is conveniently described in polar coordinates (see Figure 34) and suppose that f is continuous and

$$E = \{(x, y, z) | (x, y) \epsilon D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

Where D is given in polar coordinates by

$$D = \{(r, \theta) | \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

Then, the triple integration

$$\iiint_E f(x, y, z) dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) dz\right] dA$$

is represented by

$$\iiint_E f(x, y, z) dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta$$

Definition 5.7: The formula given by

$$\iiint_E f(x, y, z) dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta$$

is called triple integration in cylindrical coordinates.

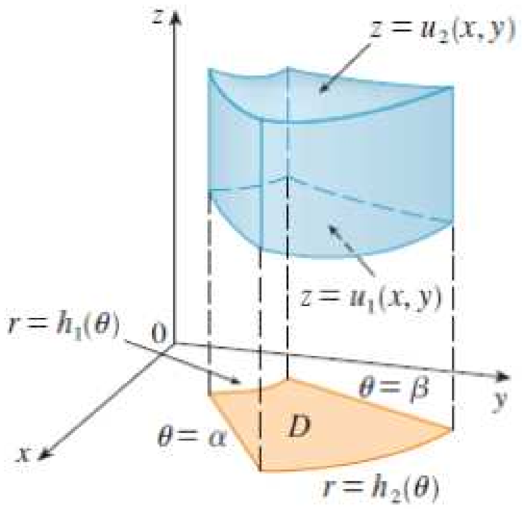

Figure 34

Example 19: A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$ (See Figure 35). The density at any point is Proportional to its distance from the axis of the cylinder. Find the mass of E.

Solution:

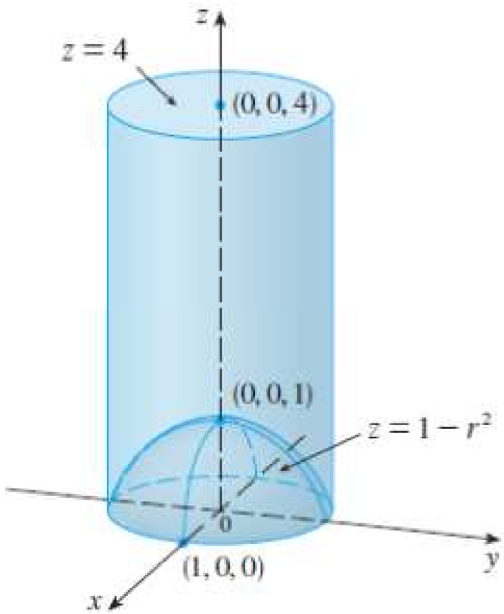

Figure 35

In cylindrical coordinates the cylinder is $r = 1$ and the paraboloid is $z = 1 - r^2$, so we can write

$$E = \{(r, \theta, z) | 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 1 - r^2 \leq z \leq 4\}$$

Since the density at (x, y, z) is proportional to the distance from the z −axis, the density function is given by

$$\rho(x, y, z) = K\sqrt{x^2 + y^2} = Kr$$

Where,K is the proportionality constant. Therefore, the mass of E is

$$m = \iiint_E K\sqrt{x^2 + y^2}dV$$

$$= \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (Kr)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 Kr^2[4 - (1 - r^2)]dr\, d\theta$$

$$= K \int_0^{2\pi} d\theta \int_0^1 (3r^2 + r^4)\, dr$$

$$= 2\pi K \left[r^3 + \frac{r^5}{5}\right]_0^1 = \frac{12\pi K}{5}$$

Hence, $m = \frac{12\pi K}{5}$

5.6.2. Spherical coordinates

In spherical coordinates we represent a point P by order triple $(\rho, \theta, \emptyset)$ as shown in figure 36, where ρ is the distance from the origin to the point P, θ is the same angle as in cylindrical coordinates and $\emptyset$ is the angle between the positive z −axis and the line segment OP with $\rho \geq 0$ and $0 \leq \emptyset \leq \pi$.

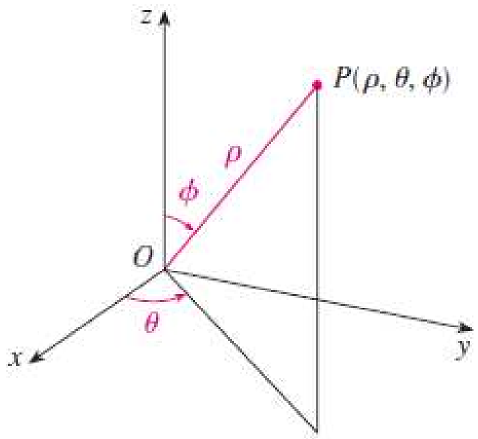

Figure 36 The spherical coordinates of a point

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point. That is when we represent regions like spheres and cones.

The relationship between rectangular and spherical coordinates can be seen from Figure37. and we have

$$z = \rho \cos \emptyset \qquad\qquad r = \rho \sin \emptyset$$

But, $x = r \cos \theta$ and $y = r \sin \theta$

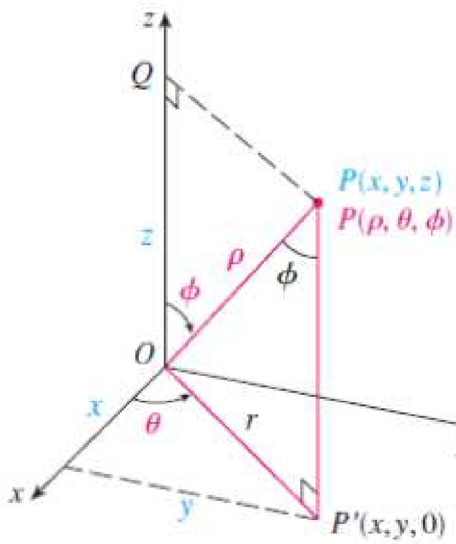

Figure 37

To convert from spherical to rectangular coordinates we use the equations

$$x = \rho \sin \emptyset \cos \theta \qquad y = \rho \sin \emptyset \sin \theta \qquad z = \rho \cos \emptyset$$

To convert from rectangular to spherical coordinates we use the equations

$$\rho^2 = x^2 + y^2 + z^2 \qquad \tan \theta = \frac{y}{x} \qquad \cos \emptyset = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

Example 20: (a) The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates, then find its rectangular coordinate.

(b) The point $(0,2\sqrt{3},-2)$ is given in rectangular coordinates. Find the spherical coordinates of this point.

Solution: (a) the rectangular coordinates are

$$x = \rho \sin \emptyset \cos \theta = 2 \sin \frac{\pi}{3} \cos \frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin \emptyset \sin \emptyset = 2 \sin \frac{\pi}{3} \sin \frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos \emptyset = 2 \cos \frac{\pi}{3} = 2\left(\frac{1}{2}\right) = 1$$

Hence, the point in rectangular coordinate is $\left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\right)$

(b) The cylindrical coordinates are

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0 + \left(2\sqrt{3}\right)^2 + (-2)^2} = 4$$

$$\cos \emptyset = \frac{z}{\rho} = \frac{-2}{4} = \frac{-1}{2}, \ \emptyset = \frac{2\pi}{3}$$

$$\cos \theta = \frac{x}{\rho \sin \emptyset} = 0, \ \theta = \frac{\pi}{2}$$

Thus, the point in spherical coordinate is $(4, \pi/2, 2\pi/3)$.

Triple integration in spherical coordinate

In the spherical coordinate system the rectangular box is a **spherical wedge**

$$E = \{(\rho, \theta, \emptyset) | a \le \rho \le b, \alpha \le \theta \le \beta, c \le \emptyset \le d\}$$

Where, $a \ge 0, \beta - \alpha \le 2\pi, d - c \le \pi$. Now divide E into small spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half – planes $\theta = \theta_j$ and half – cones $\emptyset = \emptyset_k$.

So, from figure 38 an approximation to the volume of E_{ijk} is given by

$$\Delta V_{ijk} \approx (\Delta\rho)(\rho_i \Delta\emptyset)(\rho_i \sin\emptyset_k \Delta\theta) = {\rho_i}^2 \sin\emptyset_k \Delta\rho\Delta\theta\Delta\emptyset$$

With the aid of the Mean Value Theorem the volume of E_{ijk} is given exactly by

$$\Delta V_{ijk} = {\tilde{\rho}_i}^2 sin\tilde{\emptyset}_k \Delta\rho\Delta\theta\Delta\emptyset$$

Where, $(\tilde{\rho}_i, \tilde{\theta}_j, \tilde{\emptyset}_k)$ is a point in E_{ijk} and let $({x_{ijk}}^*, {y_{ijk}}^*, {z_{ijk}}^*)$ be the rectangular coordinates of this point, then

$$\iiint_E f(x, y, z) dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f({x_{ijk}}^*, {y_{ijk}}^*, {z_{ijk}}^*) \Delta V_{ijk}$$

$$= \lim_{l,m,n\to\infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\tilde{\rho}_i sin\tilde{\emptyset}_k cos\tilde{\theta}_j, \tilde{\rho}_i sin\tilde{\emptyset}_k sin\tilde{\theta}_j, \tilde{\rho}_i cos\tilde{\emptyset}_k)\, {\tilde{\rho}_i}^2 sin\tilde{\emptyset}_k \Delta\rho\Delta\theta\Delta\emptyset$$

Finally, if we evaluate the above limit then we get the following formula.

Definition 5.8: The formula given by

$\iiint_E f(x, y, z) dV =$

$\int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\emptyset \cos\theta, \rho \sin\emptyset \sin\theta, \rho \cos\emptyset) \rho^2 \sin\emptyset d\rho\, d\theta\, d\emptyset$,Where E is spherical wedge given by

$E = \{(\rho, \theta, \emptyset) | a \le \rho \le b, \alpha \le \theta \le \beta, c \le \emptyset \le d\}$ is called triple integration in spherical coordinates.

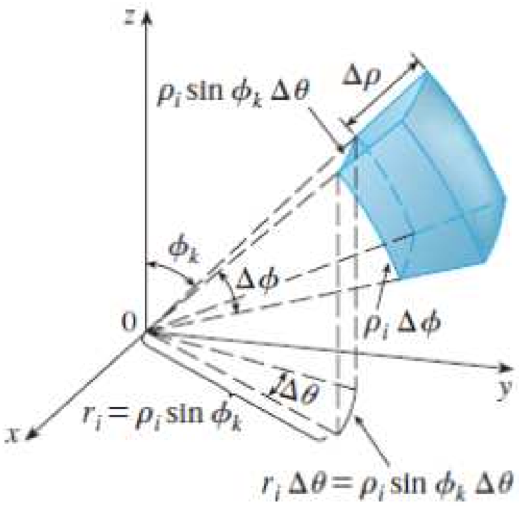

Figure 38

Example 21: Evaluate $\iiint_B \; e^{\sqrt{(x^2+y^2+z^2)^3}}\, dV$, where B is the unit ball

$$B = \{(x, y, z)| x^2 + y^2 + z^2 \leq 1\}$$

Solution: Since the boundary of is a sphere, we use spherical coordinates:

$$B = \{(\rho, \theta, \emptyset)| 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \emptyset \leq \pi\}$$

With $\rho^2 = x^2 + y^2 + z^2$, therefore,

$$\iiint_B \; e^{\sqrt{(x^2+y^2+z^2)^3}}\, dV = \int_0^{\pi} \int_0^{2\pi} \int_0^{1} e^{\sqrt{(\rho^2)^3}} \rho^2 \sin\emptyset d\rho \; d\theta \; d\emptyset$$

$$= \int_0^{\pi} \sin\emptyset d\emptyset \int_0^{2\pi} d\theta \int_0^{1} \rho^2 e^{\rho^3} d\rho$$

$$= [-\cos\emptyset]_0^{\pi} (2\pi) \left[\frac{1}{3} e^{\rho^3}\right]_0^1$$

$$= \frac{4}{3}\pi(e - 1)$$

Exercise 5.6

1. *Evaluate the cylindrical coordinate integrals of the following.*

   a. $\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}(r^2\sin^2\theta + z^2)\,dz\,r\,dr\,d\theta$

   b. $\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} z\,dz\,r\,dr\,d\theta$

   c. $\int_0^{2\pi}\int_0^{\theta/2\pi}\int_0^{3+24r^2} dz\,r\,dr\,d\theta$

   d. $\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\,r\,dr\,d\theta$

   e. $\int_0^{2\pi}\int_0^3\int_{r^2/3}^{\sqrt{18-r^2}} dz\,r\,dr\,d\theta$

2. *Write the following equations in spherical coordinates.*

   a. $x^2+y^2+(z-1)^2=1$

   b. $z=\sqrt{x^2+y^2}$

   c. $z^2=x^2+y^2$

   d. $x^2+z^2=9$

   e. $x^2-2x+y^2+z^2=0$

   f. $x+2y+3z=1$

3. *Evaluate the spherical coordinate integrals of the following.*

   a. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\emptyset}(\rho\cos\emptyset)\rho^2\sin\emptyset\;d\rho\,d\emptyset\;d\theta$

   b. $\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\emptyset}\rho^2\;\sin\emptyset\,d\rho\,d\emptyset\,d\theta$

   c. $\int_0^{3\pi/2}\int_0^{\pi}\int_0^1 5\rho^3\;\sin^3\emptyset\,d\rho\,d\emptyset\,d\theta$

   d. $\int_0^{2\pi}\int_0^{\pi/4}\int_0^2(\rho\cos\emptyset)\rho^2\sin\emptyset\;d\rho\,d\emptyset\;d\theta$

   e. $\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\emptyset}^2 3\rho^2\sin\emptyset\,d\rho\,d\emptyset\,d\theta$

5.7. Applications of triple integrals

Overview

In this section we will see the different applications of triple integrals, such as volumes, mass and center of mass of different solid regions.

Section objective:

After the completion of this section, successful students be able to:

- Apply triple integrals
- Evaluate different examples on the applications

5.7.1. Volume

Recall that if $f(x) \geq 0$, then the single integral $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b, and if $f(x,y) \geq 0$, then the double integral $\iint_D f(x,y)dA$ represents the volume under the surface $z = f(x,y)$ and above D . The corresponding interpretation of a triple integral $\iiint_E f(x,y,z)dV$, where $f(x,y,z) \geq 0$ is not very useful because it would be a four dimensional object and that is very difficult to visualize.
But with special case where $f(x,y,z) = 1$, for all points in E, then the triple integral represents the volume of E.

$$V(E) = \iiint_E dV$$

Example 22: Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y, x = 0$ *and* $z = 0$.

Solution: The tetrahedron T and its projection D on the xy −plane are shown in Figures 39 and 40. The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane

$x + 2y + z = 2$ or $z = 2 - x - 2y$.

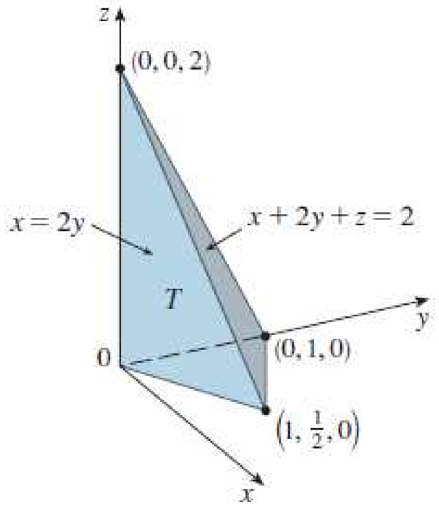

Figure 39

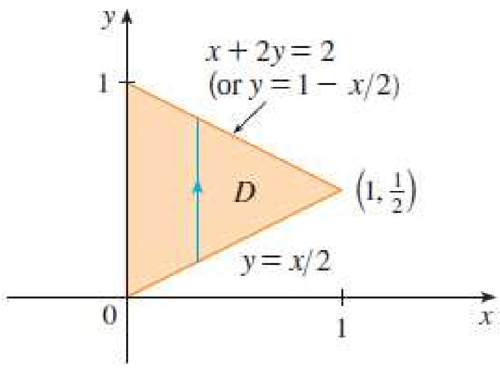

Figure 40

Therefore, we have

$$V(T) = \iiint_T \, dV = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\, dy\, dx$$

$$V(T) = \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y) dy\, dx = \frac{1}{3}$$

$$\Rightarrow V(T) = \frac{1}{3}$$

5.7.2. Center of mass in triple integrals

Definition 5.9: If the density function of a solid object that occupies the region E is (x, y, z) , in units of mass per unit volume, at any given point (x, y, z) , then its **mass** is

$$m = \iiint_E \rho(x, y, z) dv$$

and its **moments** about the three coordinate planes are

$$M_{yz} = \iiint_E x\rho(x,y,z)dV$$

$$M_{xz} = \iiint_E y\rho(x,y,z)dV$$

$$M_{xy} = \iiint_E z\rho(x,y,z)dV$$

Then, the center of mass is at the point $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

Example 23: Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z, z = 0$ and $x = 1$.

Solution: The solid E and its projection onto the xy −plane are shown in Figures 41 and 42. The lower and upper surfaces of E are the planes $z = 0$ & $z = x$, so we describe E as a type 1 region:

$$E = \{(x,y,z)|-1 \le y \le 1, y^2 \le x \le 1, 0 \le z \le x\}$$

Then, if the density is $\rho(x,y,z) = \rho$, the mass is

$$m = \iiint_E \rho dV = \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{x} \rho \, dz \, dx \, d\, y$$

$$= \rho \int_{-1}^{1}\int_{y^2}^{1} x \, dx \, dy = \rho \int_{-1}^{1} \left[\frac{x^2}{2}\right]_{y^2}^{1} dy$$

$$= \frac{\rho}{2}\int_{-1}^{1}(1 - y^4)dy = \rho \int_{0}^{1}(1 - y^4)dy$$

$$= \rho \left[y - \frac{y^5}{5}\right]_0^1 = \frac{4\rho}{5}$$

Because of the symmetry of E and ρ about the xz −plane, we can immediately say that $M_{xz} = 0$ and, therefore $\bar{y} = 0$.

$$M_{yz} = \iiint_E \ x\rho \, dV = \int_{-1}^{1}\int_{y^2}^{1}\int_0^x x\rho \, dz \, dx \, dy$$

$$= \rho \int_{-1}^{1}\int_{y^2}^{1} x^2 dx \, dy = \rho \int_{-1}^{1}\left[\frac{x^3}{3}\right]_{y^2}^{1} dy$$

$$= \frac{2\rho}{3}\int_0^1 (1 - y^6) \, dy = \frac{2\rho}{3}\left[y - \frac{y^7}{7}\right]_0^1$$

$$= \frac{4\rho}{7}$$

$$M_{xy} = \iiint_E \ z\rho \, dV = \int_{-1}^{1}\int_{y^2}^{1}\int_0^x z\rho \, dz \, dx \, dy$$

$$= \rho \int_{-1}^{1}\int_{y^2}^{1}\left[\frac{z^2}{2}\right]_0^x dx \, dy = \frac{\rho}{2}\int_{-1}^{1}\int_{y^2}^{1} x^2 dx \, dy$$

$$= \frac{\rho}{3}\int_0^1 (1 - y^6) dy$$

$$= \frac{2\rho}{7}$$

Therefore the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \left(\frac{4\rho/7}{4\rho/5}, 0, \frac{2\rho/7}{4\rho/5}\right)$$

$$= \left(\frac{5}{7}, 0, \frac{5}{14}\right)$$

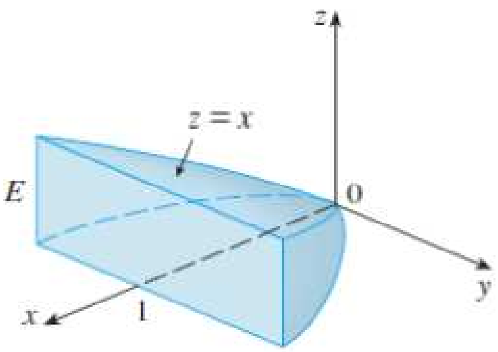

Figure 41

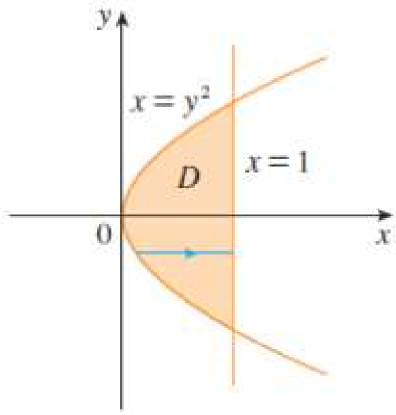

Figure 42

Exercise 5.7

1. *Use a triple integral to find the volume of the following solid regions.*
   a. *The tetrahedron enclosed by the coordinate planes and the plane* $2x + y + z = 4$.
   b. *The solid enclosed by the paraboloid* $x = y^2 + z^2$ *and the plane* $x = 16$.
   c. *The rectangular solid in the first octant bounded by the coordinate planes and the planes* $x = 1, y = 2$ *and* $z = 3$.
   d. *The region bounded by the paraboloids* $z = 8 - x^2 - y^2$ *and* $z = x^2 + y^2$.
   e. *The region in the first octant enclosed by the cylinder* $x^2 + z^2 = 4$ *and the plane* $y = 3$.
2. *Find the mass and center of mass of the solid* E *with the given density function* ρ.
   a. E *is the tetrahedron bounded by the planes* $x = 0, y = 0, z = 0, x + y + z = 1$; $\rho(x, y, z) = y$.
   b. E *is the cube given by* $0 \le x \le a, 0 \le y \le a, 0 \le z \le a$; $\rho(x, y, z) = x^2 + y^2 + z^2$.
   c. E *is a solid bounded by the parabolic cylinder* $z = 1 - y^2$ *and the planes* $x + z = 1, x = 0$ *and* $z = 0$; $\rho(x, y, z) = 4$.
   d. E *is a solid bounded below by the disk* $R: x^2 + y^2 \le 4$ *in the plane* $z = 0$ *and above by the paraboloid* $z = 4 - x^2 - y^2$; $\rho(x, y, z) = \rho(constant)$.

e. E *is a solid bounded below by the surface* $z = 4y^2$, *above by the plane* $z = 4$, *and an the ends by the planes* $x = 1$ *and* $x = -1$: $\rho(x, y, z) = \rho(constant)$.

f. E *is a solid bounded below by the plane* $z = 0$, *on the sides by the elliptical cylinder* $x^2 + 4y^2 = 4$ *and above by the plane* $z = 2 - x$; $\rho(x, y, z) = \rho(constant)$.

g. E *is a solid bounded below by the paraboloid* $z = x^2 + y^2$ *and above by the plane* $z = 4$; $\rho(x, y, z) = \rho(constant)$.

Unit Summary:

- If f is defined on a closed, bounded rectangular region R in the xy plane, then the double integral of f over R is given by

  $$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{k=1}^{m}\sum_{k=1}^{n} f(x_k{}^*, y_k{}^*)$$

  But if R is given by $[a,b]\times[c,d]$, then the above limit is approximated by

  $$\lim_{m,n\to\infty} \sum_{k=1}^{m}\sum_{k=1}^{n} f(x_k{}^*, y_k{}^*) = \int_c^d \int_a^b f(x,y)dxdy$$

  Therefore, $\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy$

- If f is continuous on a type I region D with

  $D = \{(x,y)|a \le x \le b, g_1(x) \le y \le g_2(x)\}$, then

  $$\iint_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

- If f is continuous on a type I region D with

  $D = \{(x,y)|c \le x \le d, h_1(y) \le x \le h_2(y)\}$, then

- $\iint_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy$
- The polar coordinates (r,θ) of a point are related to the rectangular coordinates (x,y) by

  $r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$

- If f is continuous on a polar rectangle R given by $0 \le a \le r \le b, \alpha \le \theta \le \beta$ with $0 \le \beta - \alpha \le 2\pi$, then

  $$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)r\,dr\,d\theta$$

- The triple integral of f over the region or box B is given by

  $$\iiint_B f(x,y,z)dV = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}{}^*, y_{ijk}{}^*, z_{ijk}{}^*)$$

  If B is given by $B = [a,b]\times[c,d]\times[r,s]$, the limit is approximated by

  $$\lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}{}^*, y_{ijk}{}^*, z_{ijk}{}^*) = \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz$$

  Therefore, $\iiint_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz$

- To convert from cylindrical to rectangular coordinates, we use the relation

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

And to convert from rectangular to cylindrical coordinates, we use

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

- The triple integration in cylindrical coordinates is given by

$$\iiint_E f(x,y,z)dV = \int_{\alpha}^{\beta}\int_{h_1(\theta)}^{h_2(\theta)}\int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta\ , z)r\ dzdrd\theta$$

- To convert from spherical to rectangular coordinates, we use the relation

$$x = \rho\sin\emptyset\cos\theta \qquad y = \rho\sin\emptyset\sin\theta \qquad z = \rho\cos\emptyset$$

And to convert from rectangular to cylindrical coordinates, we use

$$\rho^2 = x^2 + y^2 + z^2 \qquad \tan\theta = \frac{y}{x} \qquad \cos\emptyset = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

- The triple integration in cylindrical coordinates is given by

$$\iiint_E f(x,y,z)dV =$$

$$\int_c^d\int_{\alpha}^{\beta}\int_a^b f(\rho\sin\emptyset\cos\theta\ , \rho\sin\emptyset\sin\theta\ , \rho\cos\emptyset)\rho^2\sin\emptyset\ \ d\rho d\theta d\emptyset$$

Miscellaneous Exercises

1. Evaluate $\iint_R (1-6x^2y)dA$, where $R=\{(x,y)|\ 0\le x\le 2, -1\le y\le 1\}$.
2. Find the volume of the region bounded by the paraboloid $z=x^2+y^2$ and below by the triangle enclosed by the lines $y=x, x=0$ and $x+y=2$ in the xy plane.
3. Evaluate $\iint_r xy\, dA$, where R is the region bounded by the lines $y=x, y=2x$ and $x+y=2$.
4. Find the volume of the solid whose base is the region in the xy plane that is bounded by the parabola $y=4-x^2$ and the line $y=3x$, while the top of the solid is bounded by the plane $z=x+4$.
5. Calculate $\iint_R \frac{\sin x}{x}\, dA$, where R is the triangle in the xy plane bounded by the x −axis, the line $y=x$ and the line $x=1$.
6. Evaluate the following integrals by converting them into polar coordinates
   a. $\iint_D 2xy\, dA$, where D is the portion of the region between the circles of radius 2 and radius 5 centered at the origin that lies in the first quadrant.
   b. $\iint_D e^{x^2+y^2}dA$, where D is the unit circle centered at the origin.
7. Find the area of the region R enclosed by the parabola $y=x^2$ and the line $y=x+2$.
8. Find the mass and center of mass of a thin plate of density $\rho=3$ bounded by the lines $x=0, y=x$ and the parabola $y=2-x^2$ in the first quadrant.
9. Find the area of the region bounded by the parabolas $x=y^2-1$ and $x=2y^2-2$.
10. Find the mass and center of mass about the x −axis of a thin plate bounded by the curves $x=y^2$ and $x=2y-y^2$ if the density at the point (x,y) is $\rho(x,y)=y+1$
11. Evaluate $\iiint_E x\, dV$, where E is the solid region bounded by the cylinder $x^2+y^2=4$ and the plane $2y+z=4$.
12. Using spherical coordinates evaluate $\iiint_E 16z\, dV$, where E is the upper half of the sphere $x^2+y^2+z^2=1$.

13. Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.
14. Find the mass and center of mass of a solid of constant density bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$.
15. A solid of constant density is bounded below by the plane $z = 0$, on the sides by the elliptical cylinder $x^2 + 4y^2 = 4$ and above by the plane $z = 2 - x$, then find the mass and center of mass of the solid.

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